0. CHAPTER 0 : Preface to the Student

0.0.1. **Remark** (Course Goals).
Our main goals in this course are:

- to improve our ability to think and reason (useful no matter what career you choice)
- to write in a professional mathematical fashion
- to gain a solid understand of the material most useful for advanced math courses.
- Do you want to add more?

Although you might not become a research mathematician, in almost any mathematically related work you may do, the kind of reasoning you need to be able to do is the same reasoning you use in proving theorems. You must first understand exactly what you want to prove (verify, show, or explain), and you must be familiar with the logical steps that allow you to get from the hypothesis to the conclusion.

0.0.2. **Remark.** Various sets of numbers are excellent sources for developing an understanding of the structure of a correct proof. So the following definitions and remarks, which you may use when forming a proof, will be used extensively in early examples of proof writing.

0.0.3. **Definition** (Sets and Numbers and their Symbols).

real numbers	=	$\mathbb{R}$		
natural numbers	=	$\mathbb{N}$	=	$\{1, 2, 3, 4, \ldots\}$
integers	=	$\mathbb{Z}$	=	$\{0, \pm1, \pm2, \pm3, \pm4 \ldots\}$
rational numbers	=	$\mathbb{Q}$	=	$\left\{\frac{p}{q} \in \mathbb{R} \colon p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
irrational numbers	=	$\mathbb{R} \setminus \mathbb{Q}$	=	$\{x \colon x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$
complex numbers	=	$\mathbb{C}$	=	$\{a + ib \colon a, b \in \mathbb{R}\}$
positive real numbers	=	$\mathbb{R}^{>0}$	=	$\{x \in \mathbb{R} \colon x > 0\}$
nonnegative real numbers	=	$\mathbb{R}^{\geq 0}$	=	$\{x \in \mathbb{R} \colon x \geq 0\}$
negative real numbers	=	$\mathbb{R}^{<0}$	=	$\{x \in \mathbb{R} \colon x < 0\}$
nonpositive real numbers	=	$\mathbb{R}^{\leq 0}$	=	$\{x \in \mathbb{R} \colon x \leq 0\}$
prime numbers	=	$\mathbb{P}$		$\{p \in \mathbb{N} \colon p \neq 1$ and the only natural numbers that divide p are 1 and $p\}$
composite numbers	=		=	$\{n \in \mathbb{N} \colon n \neq 1 \text{ and } n \notin \mathbb{P}\} = \mathbb{N} \setminus (\{1\} \cup \mathbb{P})$
empty set	=	$\emptyset$	=	a set with no elements
a finite set	=		=	an empty set or a set with $n \in \mathbb{N}$ elements
an infinte set	=		=	a set that is not a finite set

0.0.4. **Definition** (More on Numbers).

<u>Even/Odd Numbers</u>

(1) $x \in \mathbb{Z}$ is **even** if and only if there is $k \in \mathbb{Z}$ such that $x = 2k$ $\iff$ $x \in 2\mathbb{Z}$.
(1') $x \in \mathbb{N}$ is **even** if and only if there is $k \in \mathbb{N}$ such that $x = 2k$ $\iff$ $x \in 2\mathbb{N}$.
(2) $x \in \mathbb{Z}$ is **odd** if and only if there is $j \in \mathbb{Z}$ such that $x = 2j - 1$ $\iff$ $x \in 2\mathbb{Z} - 1$.
(2') $x \in \mathbb{Z}$ is *odd* if and only if there is $j \in \mathbb{Z}$ such that $x = 2j + 1$ $\iff$ $x \in 2\mathbb{Z} + 1$.
(2") $x \in \mathbb{N}$ is **odd** if and only if there is $j \in \mathbb{N}$ such that $x = 2j - 1$ $\iff$ $x \in 2\mathbb{N} - 1$.
(2"') $x \in \mathbb{N}$ is *odd* and $x \geq 3$ if and only if there is $j \in \mathbb{N}$ such that $x = 2j + 1$.

<u>Divides</u> $a|b \iff a$ divides $b \iff a$ is a divisor of $b \iff b$ is divisible by $a \iff b$ is a multiple of a .

(3) For $a, b \in \mathbb{N}$: a **divides** b if and only if there is a $k \in \mathbb{N}$ such that $ak = b$.
So, for $a, b \in \mathbb{N}$: $a \mid b$ if and only if $\frac{b}{a} \in \mathbb{N}$.
(3') For $a, b \in \mathbb{Z}$: a **divides** b if and only if there is a $k \in \mathbb{Z}$ s.t. $ak = b$ and $a \neq 0$.
So, for $a, b \in \mathbb{Z}$: $a \mid b$ if and only if $\frac{b}{a} \in \mathbb{Z}$.

0.0.5. **Remark** (The Natural Numbers $\mathbb{N}$).
The properties below describe the basic arithmetical and ordering stucture of the set $\mathbb{N}$ of natural numbers.

($\mathbb{N}1$) Successor properties
- 1 is a natural number
- Every natural number x has a unique successor $x+1$
- 1 is not the successor of any natural number.

($\mathbb{N}2$) Closure properties
- The sum of two natural numbers is a natural number.
- The product of two natural nubmers is a natural number.

($\mathbb{N}3$) Associativity properties
- $\forall x, y, z \in \mathbb{N}$, $\quad x + (y + z) = (x + y) + z.$
- $\forall x, y, z \in \mathbb{N}$, $\quad x(yz) = (xy)z.$

($\mathbb{N}4$) Commutativity properties
- $\forall x, y \in \mathbb{N}$, $\quad x + y = y + x.$
- $\forall x, y \in \mathbb{N}$, $\quad xy = yx.$

($\mathbb{N}5$) Distributivity properties
- $\forall x, y, z \in \mathbb{N}$, $\quad x(y + z) = xy + xz.$
- $\forall x, y, z \in \mathbb{N}$, $\quad (y + z)x = yx + zx.$

($\mathbb{N}6$) Cancellation properties
- $\forall x, y, z \in \mathbb{N}$, $\quad$ if $x + z = y + z$ the $x = y$.
- $\forall x, y, z \in \mathbb{N}$, $\quad$ if $xz = yz$ the $x = y$.

0.0.6. **Theorem** (The Fundamental Theorem of Arithmetic).
Each natural number larger than 1 is a prime number or can be expressed as a product of primes.
If we list the prime factors in increasing order, then there is only one prime factoration: the primes and their exponts are uniquely determined.
For example, $440 = 2^3 \cdot 5 \cdot 11$.

0.0.7. **Remark** (The Integers $\mathbb{Z}$).
$\mathbb{Z}$ share properties $\mathbb{N}2$-$\mathbb{N}6$ with $\mathbb{N}$ (except in $\mathbb{N}6$ we cannot cancel $z = 0$ from the product $xz = yz$).
Other important properties are:

($\mathbb{Z}7$) $\forall x \in \mathbb{Z}$: $\quad x + 0 = x$, $x \cdot 0 = 0$, $x + (-x) = 0$.
($\mathbb{Z}8$) $\forall x, y, z \in \mathbb{Z}$: $\quad$ if $x < y$ and $z > 0$, then $xz < yz$. $\quad$ typo in book here
($\mathbb{Z}9$) If $x, y \in \mathbb{Z}^{>0}$, then $xy \in \mathbb{Z}^{>0}$
If $x, y \in \mathbb{Z}^{<0}$, then $xy \in \mathbb{Z}^{>0}$
If $x \in \mathbb{Z}^{>0}$ and $y \in \mathbb{Z}^{<0}$, then $xy \in \mathbb{Z}^{<0}$
($\mathbb{Z}10$) Each integer is either even or odd, but not both.

0.0.8. **Remark** (The Real numbers $\mathbb{R}$).
We think of the real numbers as being all the numbers along the number line. The number system $\mathbb{R}$ shares many of the properties of $\mathbb{N}$ and $\mathbb{Z}$. Here are a few more properties of $\mathbb{R}$.

($\mathbb{R}11$) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$.
($\mathbb{R}12$) Each real number is either rational or irrational, but not both.
($\mathbb{R}13$) If $x \in \mathbb{R} \setminus \{0\}$, then x has a multiplicative inverse (i.e., $\exists y \in \mathbb{R}$ such that $xy = 1$).

0.0.9. **Remark** (Space for you to write more useful things as we proceed through the course).

1. CHAPTER 1: LOGIC and PROOFS

1.1. Section 1.1: Propositions and Connectives.

1.1.1. **Definition.** A **proposition** is a meaningful declarative statement that is either true or false; thus, a proposition has exactly one **truth value**: true (T) or false (F). We often represent a proposition by a letter (such a P), called a **propositional variable**, similar to the way we represent a number by a mathematical variable (such as x)

1.1.2. **Definition.** A **paradox** is a meaningful declarative statement that is neither true nor false. An example of a paradox is: *This sentence is false.*

1.1.3. **Notation. Logical Connective Symbols:**

$\wedge$	and
$\vee$	or
$\sim$	not
$\Rightarrow$	implies, if ... then
$\Leftrightarrow$	if and only if

1.1.4. **Definition.** A proposition is **simple** (or **atomic**) if no part of it is itself a proposition. A proposition is **compound** if it can be decomposed into simple propositions, called its **components**. Compound propositions can be formed by (properly) combining simple propositions and logical symbols.

1.1.5. **Definition.** A **propositional form** is an expression involving finitely many logical connective symbols (e.g.: $\wedge$, $\vee$, $\sim$) and letters (e.g.: P, Q, R) that represent propositions. If the letters (think of them as propositional variables) are replaced with propositions then the expression is a proposition.

1.1.6. **Definition.** Let P and Q be propositions.

(1) The **negation** of P, denoted $\sim P$, is the compound proposition "not P".
$\sim P$ is true exactly when P is false.
(2) The **conjunction** of P and Q, denoted $P \wedge Q$, is the compound proposition "P and Q".
$P \wedge Q$ is true exactly when both P and Q are true.
(3) The **disjunction** of P and Q, denoted $P \vee Q$, is the compound proposition "P or Q".
$P \vee Q$ is true exactly when at least one of P or Q is true.

1.1.7. **Definition.** A **truth table** of a propositional form exhibits the truth values (TorF) of the propositional form for each possible combination of truth values for its components.

1.1.8. **Truth Table** (for $\sim$, $\wedge$, $\vee$).

P	Q	$\sim$ P	P$\wedge$ Q	P $\vee$ Q
T	T	F	**T**	T
F	T	T	F	T
T	F		F	T
F	F		F	**F**

1.1.9. **Remark.** The truth table of a compound proposition form consisting of n simple proposition forms linked with connectives has 2^n lines. This exhausts all possible truth values of the simple propositions.

1.1.10. **Definition.** A **tautology** is a propositional form that is true for each assignment of truth values to its components.

1.1.11. **Definition.** A **contradiction** is a propositional form that is the negative of a tautology. Thus, a propositional form is a contradiction if and only if it is false for each assignment of truth values to its components.

1.1.12. **Remark.** Just as in algebra, we give the connectives a priority ordering that resolves ambiguities when parentheses are omitted. The **priority of the connectives**, from highest to lowest, is:
$\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

1.1.13. **Definition.**
Two *propositions* are **equivalent** if and only if they have the same truth values.
Two *propositional forms* are **equivalent** if and only if they have the same truth tables.

1.1.14. **Definition.** A **denial** of a proposition P is any proposition equivalent to $\sim P$.

1.1.15. **Remark.** $\star$ means know this one!

1.1.16. **Theorem** (Theorem 1.1.1 in book). Let P, Q, and R be propositional variables.

$\star$ a.	P	is equivalent to	$\sim(\sim P)$	(double negation law)
$\star$ b.	$P \vee Q$	is equivalent to	$Q \vee P$	(commutative law)
$\star$ c.	$P \wedge Q$	is equivalent to	$Q \wedge P$	(commutative law)
$\star$ d.	$P \vee (Q \vee R)$	is equivalent to	$(P \vee Q) \vee R$	(associative law)
$\star$ e.	$P \wedge (Q \wedge R)$	is equivalent to	$(P \wedge Q) \wedge R$	(associative law)
$\star$ f.	$[\ P \wedge (Q \vee R)\ ]$	is equivalent to	$[\ (P \wedge Q) \vee (P \wedge R)\ ]$	(distributive law)
$\star$ g.	$[\ P \vee (Q \wedge R)\ ]$	is equivalent to	$[\ (P \vee Q) \wedge (P \vee R)\ ]$	(distributive law)
$\star$h.	$[\ \sim(P \wedge Q)\ ]$	is equivalent to	$[\ (\sim P) \vee (\sim Q)\ ]$	(DeMorgan's law)
$\star$ i.	$[\ \sim(P \vee Q)\ ]$	is equivalent to	$[\ (\sim P) \wedge (\sim Q)\ ]$	(DeMorgan's law)

1.2. Section 1.2: Conditionals and Biconditionals.

1.2.1. **Definition.** Let P and Q be propositions.

(1) The **conditional sentence** $P \Rightarrow Q$, read "P implies Q", is the proposition "If P then Q".
(2) The **biconditional sentence** $P \Leftrightarrow Q$ is the proposition "P if and only if Q".

Their truth tables are given below.

1.2.2. **Truth Table** (for $\Rightarrow$, $\Leftrightarrow$)**.**

P	Q	P$\Rightarrow$ Q	P $\Leftrightarrow$ Q
T	T	T	T
F	T	T	F
T	F	F	F
F	F	T	T

1.2.3. **Remark.**
Warnings:

(1) If P is false, then $P \Rightarrow Q$ is true. Thus, think of $\Rightarrow$ as in a promise.
(2) $P \Rightarrow Q$ may be true even when there is no (everyday) connection between P and Q since the truth value of $P \Rightarrow Q$ depends *only* on the truth value of P and Q and not on the (everday) interpretation of P and Q.

Note:

(3) "if and only if" is often abbreviated to "iff"
(4) The propositional forms P and Q are *equivalent* precisely when $P \Leftrightarrow Q$ is a *tautology.*

1.2.4. **Remark.** Translations of $P \Rightarrow Q$:

(1) If P, then Q
(2) P implies Q
(3) P is sufficient for Q
(4) P only if Q
(5) P only when Q
(6) P only whenever Q
(7) Q is necessary for P
(8) Q if P
(9) Q when P
(10) Q whenever P

1.2.5. **Remark.** Translations of $P \Leftrightarrow Q$:

(1) P is equivalent to Q
(2) P if and only if Q (See IS 1.2.4, (4) and (8))
(3) P if but only if Q
(4) P precisely when Q
(5) P is necessary and sufficient for Q (See IS 1.2.4, (3) and (7))

1.2.6. **Definition.** For the conditional sentence $P \Rightarrow Q$:

(1) P is the **antecedent** and Q is the **consequent**

(2) the **converse** of $P \Rightarrow Q$ is: $Q \Rightarrow P$

(3) the **contrapositive** of $P \Rightarrow Q$ is: $(\sim Q) \Rightarrow (\sim P)$.

1.2.7. **Remark.** $\star$ means know this one!

1.2.8. **Theorem** (Theorem 1.2.1 in book)**.** Consider the propositional form $P \Rightarrow Q$.

$\star$ a.	$P \Rightarrow Q$	is equivalent to	its contrapositive	$(\sim Q) \Rightarrow (\sim P)$.
$\star$ b.	$P \Rightarrow Q$	is **not** equivalent to	its converse	$Q \Rightarrow P$

1.2.9. **Theorem** (Theorem 1.2.2 in book)**.**

$\star$ a.	$[\ P \Rightarrow Q\ ]$	is equivalent to	$[\ (\sim P) \vee Q\ ]$	
$\star$ b.	$[\ P \Leftrightarrow Q\ ]$	is equivalent to	$[\ (P \Rightarrow Q) \wedge (Q \Rightarrow P)\ ]$	(biconditional)
$\star$ c.	$[\ \sim (P \Rightarrow Q)\ ]$	is equivalent to	$[\ P \wedge \sim Q\ ]$	(denial of implies)
$\star$ d.	$[\ \sim (P \wedge Q)\ ]$	is equivalent to	$[\ P \Rightarrow \sim Q\ ]$	(denial of and)
$\star$ d.	$[\ \sim (P \wedge Q)\ ]$	is equivalent to	$[\ Q \Rightarrow \sim P\ ]$	(denial of and)
$\star$ e.	$[\ P \Rightarrow (Q \Rightarrow R)\ ]$	is equivalent to	$[\ (P \wedge Q) \Rightarrow R\ ]$	
$\star$ f.	$[\ P \Rightarrow (Q \wedge R)\ ]$	is equivalent to	$[\ (P \Rightarrow Q) \wedge (P \Rightarrow R)\ ]$	
$\star$ g.	$[\ (P \vee Q) \Rightarrow R\ ]$	is equivalent to	$[\ (P \Rightarrow R) \wedge (Q \Rightarrow R)\ ]$	

1.3. <u>Section 1.3: Quantifiers</u>.

1.3.1. Definition.

- An **open sentence** (or **predicate**) is a sentence containing one or more variables, which becomes a proposition when the variables are replaced by the names of particular objects. An open sentence containing n variables, say $x_1, x_2, \ldots, x_n$, is denoted by $P(x_1, x_2, \ldots, x_n)$.
- The **universe** (or in long **universe of discourse**) of $P(x_1, x_2, \ldots, x_n)$ is the collection of all objects from which $x_1, x_2, \ldots, x_n$ can be taken.
- The **truth set** of $P(x_1, x_2, \ldots, x_n)$ is the collection of all objects from its universe for which $P(x_1, x_2, \ldots, x_n)$ is a true proposition. In many cases, the universe is understood from the context.

1.3.2. **Definition.** With a given universe specified, *two open sentences* $P(x)$ and $Q(x)$ are **equivalent** iff they have the same truth set.

1.3.3. **Definition.** Let $P(x)$ be an open sentence with variable x.

(1) The **universal quantifier** $\forall$:
The quantified sentence $(\forall x)P(x)$ is read
"for all x, $P(x)$" (and also sometimes as, "for each x, $P(x)$").
It is true precisely when the truth set for $P(x)$ is the *entire universe.*

(2) The **existential quantifier** $\exists$:
The sentence $(\exists x)P(x)$ is read
"there exists x such that $P(x)$" (and also sometimes as, "for some x, $P(x)$").
It is true precisely when the truth set for $P(x)$ is *nonempty.*

(3) The **unique existence quantifier** $\exists!$:
The sentence $(\exists! x)P(x)$ is read
"there exists a unique x such that $P(x)$".
It is true precisely when the truth set for $P(x)$ contains *exactly one element.*

1.3.4. **Remark.** Priority (when parentheses are excluded):
$\forall$ and $\exists$ have equal priority and
come after the logical connective symbols $\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

1.3.5. **Remark.** Let $P(x)$ and $Q(x)$ be open sentences with variable x.

(1) "All $P(x)$ are $Q(x)$" is symbolized by $(\forall x)[P(x) \Rightarrow Q(x)]$.
(2) "Some $P(x)$ are $Q(x)$" is symbolized by $(\exists x)[P(x) \wedge Q(x)]$.
(3) "Every $x \in A$ has property P" is symbolized by $(\forall x)[x \in A \Rightarrow P(x)]$,
which is often shortened to $(\forall x \in A)P(x)$.
(4) "Some $x \in A$ has property P" is symbolized by $(\exists x)[x \in A \wedge P(x)]$,
which is often shortened to $(\exists x \in A)P(x)$.

1.3.6. **Recall.** The difference between a proposition and a propositional form (see IS 1.1.1 and 1.1.5).
Now: the difference between a quantified sentence and its logical form, via an example.
Let:

the universe U be all USC students

$M(x)$ be x *is a math major*

$R(x)$ be x *rocks*.

A quantified sentence is *All USC math majors rock* .
Its (quantified) logical form is $(\forall x)[M(x) \Rightarrow R(x)]$.

1.3.7. **Definition** (for Quantified Sentences).
Two quantified sentences are **equivalent for a particular universe** iff they have the same truth value in that universe. Two quantified sentences are **equivalent** iff they are equivalent in every universe.

1.3.8. **Definition** (for Quantified Logical Forms).
Two logical forms of quantified sentences are **equivalent** iff the truth of one implies the truth of the other, and conversely, for every possible meaning of the open sentences in every universe.

<u>For IS 1.3.9, 1.3.10, 1.3.11</u>: Let $P(x)$ be an open sentence with variable x.

1.3.9. **Theorem** (<u>Theorem 1.3.1 in book</u>). <u>Denials</u>

⋆ a.	$\sim (\forall x)P(x)$	is equivalent to	$(\exists x) \sim P(x)$
	$\sim [(\forall x)P(x)]$	is equivalent to	$(\exists x)[\sim P(x)]$
⋆ b.	$\sim (\exists x)P(x)$	is equivalent to	$(\forall x) \sim P(x)$
	$\sim [(\exists x)P(x)]$	is equivalent to	$(\forall x)[\sim P(x)]$

1.3.10. **Theorem** (<u>Theorem 1.3.2 in book</u>).

⋆ a.	$(\exists! x)P(x)$	$\Rightarrow$	$(\exists x)P(x)$
⋆ b.	$(\exists! x)P(x)$	is equivalent to	$(\exists x)P(x) \quad \wedge \quad (\forall y)(\forall z)(P(y) \wedge P(z) \Rightarrow y = z)$
⋆ 11d.	$(\exists! x)P(x)$	is equivalent to	$(\exists x)[P(x) \wedge (\forall y)(P(y) \Rightarrow x = y)]$

1.3.11. **Theorem** (See 1.3.10). <u>More Denials</u> (IS 1.2.9c $\sim (P \Rightarrow Q)$ is eqiv. to $P \wedge \sim Q$.)

⋆ b.	$\sim (\exists! x)P(x)$	is equivalent to	$(\forall x)[\sim P(x)] \quad \vee \quad (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$
⋆ 11d.	$\sim (\exists! x)P(x)$	is equivalent to	$(\forall x)[\sim P(x) \vee (\exists y)(y \neq x \wedge P(y))]$