

84. Let $x = 5 \sec u$. The integral is transformed into

$$25 \int \sec^3 u \, du;$$

application of Formula 28 from the endpapers and subsequent resubstitution yields the answer

$$\frac{1}{2} \{x(x^2 - 25)^{1/2} + 25 \ln|x + (x^2 - 25)^{1/2}|\} + c$$

(an extra constant in the logarithmic term has been absorbed by the constant C of integration).

85. The partial fractions decomposition of the integrand is

$$\frac{x}{x^2 + 1} = \frac{x}{(x^2 + 1)^2},$$

and integration of these terms presents no difficulties.

86. Because $6x - x^2 = 9 - (x - 3)^2$, we use the substitution $x = 3 + 3 \sin u$. This transforms the integral into

$$\begin{aligned} \frac{1}{3} \int \frac{1}{1 + \sin u} \, du &= \frac{1}{3} \int \frac{1 - \sin u}{\cos^2 u} \, du \\ &= \frac{1}{3} \int \left(\sec^2 u - \frac{\sin u}{\cos^2 u} \right) \, du = \frac{1}{3} \left(\tan u - \frac{1}{\cos u} \right) + c \\ &= \frac{-1 + \sin u}{3 \cos u} + c = \frac{x - 6}{3(6x - x^2)^{1/2}} + c. \end{aligned}$$

87. Let $x = 2 \tan \xi$; this substitution transforms the integral into

$$\int \left(\frac{1}{2} \cos u + \frac{3}{2} \sin u \right) \, du,$$

and the rest is routine.

88. Use integration by parts with $u = \ln x$, $dv = x^{3/2} dx$. The antiderivative is

$$\frac{2}{25} x^{5/2} (-2 + 5 \ln x) + C.$$

89. If $u = 1 + \sin^2 x$, then the integral becomes

$$\frac{1}{2} \int \sqrt{u} du,$$

and there are no further difficulties.

90. Let $u = \sqrt{\sin x}$. The integral becomes

$$\int 2e^u du = 2e^u + C = 2 \exp(\sqrt{\sin x}) + C.$$

91. By parts: A good choice is to let $u = x$ and $dv = e^x \sin x dx$. It turns out that one must antidifferentiate both $e^x \sin x$ and $e^x \cos x$, but here Formulas 67 and 68 of the endpapers may be used, or integration by parts will suffice for each.

92. By parts: Let $u = x^{3/2}$, $dv = x^{1/2} \exp(x^{3/2}) dx$. The antiderivative is

$$\frac{2}{3} (x^{3/2} - 1) \exp(x^{3/2}) + C.$$

93. Let $u = \arctan x$, $dv = (x - 1)^{-3} dx$. Next, after the integration by parts, one confronts the integral

$$\frac{1}{2} \int \frac{1}{(1 + x^2)(x - 1)^2} dx.$$

The partial fractions decomposition of the integrand is

$$\frac{1}{2} \left(\frac{x}{1+x^2} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right).$$

The rest is routine.

94. Use integration by parts, with $u = \ln(1 + \sqrt{x})$ and $dv = dx$. If you choose $v = x - 1$, certain difficulties are skirted, and the resulting antiderivative is

$$(x - 1) \ln(1 + \sqrt{x}) - \frac{1}{2}x + \sqrt{x} + C.$$

95. Because $3 + 6x - 9x^2 = 4 - (3x - 1)^2$, we use the substitution

$$x = \frac{1}{3}(1 + 2 \sin u).$$

The integral is thereby transformed into

$$\frac{1}{9} \int (11 + 4 \sin u) du,$$

and the rest is standard.

96. Let $u = \tan \frac{\xi}{2}$. The integral becomes

$$\begin{aligned} \int \frac{2}{u^2 + 4u + 3} du &= \int \left(\frac{1}{u+1} - \frac{1}{u+3} \right) du \\ &= \ln \left| \frac{u+1}{u+3} \right| + C = \ln \left| \frac{1 + \tan(\xi/2)}{3 + \tan(\xi/2)} \right| + C. \end{aligned}$$

The transformations shown in the solution of Problem 16 of Section 9.8 allow you to write this answer in the form

$$\ln \left| \frac{1 + \sin \xi + \cos \xi}{3 + \sin \xi + 3 \cos \xi} \right| + C.$$

97. Multiply numerator and denominator by $\cos \xi + 1$. Because

$\cos^2 \xi - 1 = -\sin^2 \xi$, the integral becomes

$$\begin{aligned} & \int (-\sin \xi \cos \xi - \sin \xi) d\xi \\ &= \frac{1}{2} \cos^2 \xi + \cos \xi + C. \end{aligned}$$

98. Use integration by parts with $u = \tan^{-1} \sqrt{x}$ and $dv = x^{3/2} dx$. Result:

$$\begin{aligned} & \frac{2}{5} x^{5/2} \tan^{-1} \sqrt{x} - \frac{1}{5} \int \frac{x^2 - 1 + 1}{x + 1} dx \\ &= \frac{2}{5} x^{5/2} \tan^{-1} \sqrt{x} - \frac{1}{10} x^2 + \frac{1}{5} x - \frac{1}{5} \ln|x + 1| + C. \end{aligned}$$

99. Use integration by parts with $u = \sec^{-1} \sqrt{x}$, $dv = dx$.

100. Let $u = x^2$. The integral becomes

$$\frac{1}{2} \int \left(\frac{1 - u}{1 + u} \right)^{1/2} du.$$

Now let $v^2 = \frac{1 - u}{1 + u}$; then $u = \frac{1 - v^2}{1 + v^2}$ and

$$du = -\frac{4v}{(1 + v^2)^2} dv.$$

The integral is thereby converted into

$$-\frac{1}{2} \int \frac{4v^2}{(1 + v^2)^2} dv.$$

Finally let $v = \tan z$. Then we obtain

$$-2 \int \sin^2 z \, dz = \sin z \cos z - z + C.$$

Subsequent resubstitutions and simplifications lead to the answer:

$$\frac{1}{2} (1 - x^4)^{1/2} - \tan^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)^{1/2} + C.$$

101. The area is

$$\begin{aligned} A &= \int_0^1 2\pi \cosh^2 x \, dx \\ &= 2\pi \left(\frac{1}{4} \sinh 2 + \frac{1}{2} - \frac{1}{4} \sinh 0 - 0 \right) \\ &= \frac{\pi}{4} \left(e^2 - \frac{1}{e^2} + 4 \right) \sim 8.83865. \end{aligned}$$

102. The length of the curve is

$$L = \int_0^1 (1 + e^{-2x})^{1/2} \, dx.$$

Now let $e^{-x} = \tan u$. Then

$$\begin{aligned} L &= \int_{x=0}^1 - \frac{\sec^3 u}{\tan u} \, du \\ &= - \int_{x=0}^1 (\csc u + \sec u \tan u) \, du \\ &= \left[-\ln |\csc u - \cot u| - \sec u \right]_{x=0}^1 \end{aligned}$$