84. Let $x = 5 \sec u$. The integral is transformed into

$$25 \int \sec^3 u \, du;$$

application of Formula 28 from the endpapers and subsequent resubstitution yields the answer

$$\frac{1}{2} \{x(x^2 - 25)^{1/2} + 25 \ln|x + (x^2 - 25)^{1/2}|\} + C$$

(an extra constant in the logarithmic term has been absorbed by the constant C of integration).

85. The partial fractions decomposition of the integrand is

$$\frac{x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2},$$

and integration of these terms presents no difficulties.

86. Because $6x - x^2 = 9 - (x - 3)^2$, we use the substitution $x = 3 + 3 \sin u$. This transforms the integral into

$$\frac{1}{3} \int \frac{1}{1 + \sin u} du = \frac{1}{3} \int \frac{1 - \sin u}{\cos^2 u} du$$

$$= \frac{1}{3} \int (\sec^2 u - \frac{\sin u}{\cos^2 u}) du = \frac{1}{3} (\tan u - \frac{1}{\cos u}) + C$$

$$= \frac{-1 + \sin u}{3 \cos u} + C = \frac{x - 6}{3(6x - x^2)^{1/2}} + C.$$

87. Let $x = 2 \tan \xi$; this substitution transforms the integral into

$$\int (\frac{1}{2} \cos u + \frac{3}{2} \sin u) \, du,$$

and the rest is routine.

88. Use integration by parts with $u = \ln x$, $dv = x^{3/2}\,dx$. The antiderivative is

$$\frac{2}{25}x^{5/2}(-2 + 5\ln x) + C.$$

89. If $u = 1 + \sin^2 x$, then the integral becomes

$$\frac{1}{2}\int \sqrt{u}\,du,$$

and there are no further difficulties.

90. Let $u = \sqrt{\sin x}$. The integral becomes

$$\int 2e^u\,du = 2e^u + C = 2\exp(\sqrt{\sin x}) + C.$$

91. By parts: A good choice is to let $u = x$ and $dv = e^x \sin x\,dx$. It turns out that one must antidifferentiate both $e^x \sin x$ and $e^x \cos x$, but here Formulas 67 and 68 of the endpapers may be used, or integration by parts will suffice for each.

92. By parts: Let $u = x^{3/2}$, $dv = x^{1/2}\exp(x^{3/2})\,dx$. The antiderivative is

$$\frac{2}{3}(x^{3/2} - 1)\exp(x^{3/2}) + C.$$

93. Let $u = \arctan x$, $dv = (x - 1)^{-3}\,dx$. Next, after the integration by parts, one confronts the integral

$$\frac{1}{2}\int \frac{1}{(1 + x^2)(x - 1)^2}\,dx.$$

The partial fractions decomposition of the integrand is

$$\frac{1}{2}\left(\frac{x}{1 + x^2} - \frac{1}{x - 1} + \frac{1}{(x - 1)^2}\right).$$

The rest is routine.

94. Use integration by parts, with $u = \ln(1 + \sqrt{x})$ and $dv = dx$. If you choose $v = x - 1$, certain difficulties are skirted, and the resulting antiderivative is

$$(x - 1)\ln(1 + \sqrt{x}) - \frac{1}{2}x + \sqrt{x} + C.$$

95. Because $3 + 6x - 9x^2 = 4 - (3x - 1)^2$, we use the substitution

$$x = \frac{1}{3}(1 + 2\sin u).$$

The integral is thereby transformed into

$$\frac{1}{9}\int (11 + 4\sin u)\,du,$$

and the rest is standard.

96. Let $u = \tan\frac{\xi}{2}$. The integral becomes

$$\int \frac{2}{u^2 + 4u + 3}\,du = \int \left(\frac{1}{u + 1} - \frac{1}{u + 3}\right) du$$

$$= \ln\left|\frac{u + 1}{u + 3}\right| + C = \ln\left|\frac{1 + \tan(\xi/2)}{3 + \tan(\xi/2)}\right| + C.$$

The transformations shown in the solution of Problem 16 of Section 9.8 allow you to write this answer in the form

$$\ln\left|\frac{1 + \sin\xi + \cos\xi}{3 + \sin\xi + 3\cos\xi}\right| + C.$$

97. Multiply numerator and denominator by $\cos\xi + 1$. Because

$\cos^2\xi - 1 = -\sin^2\xi$, the integral becomes

$$\int (-\sin\xi\cos\xi - \sin\xi)\,d\xi$$

$$= \frac{1}{2}\cos^2\xi + \cos\xi + C.$$

98. Use integration by parts with $u = \tan^{-1}\sqrt{x}$ and $dv = x^{3/2}\,dx$. Result:

$$\frac{2}{5}x^{5/2}\tan^{-1}\sqrt{x} - \frac{1}{5}\int \frac{x^2 - 1 + 1}{x + 1}\,dx$$

$$= \frac{2}{5}x^{5/2}\tan^{-1}\sqrt{x} - \frac{1}{10}x^2 + \frac{1}{5}x - \frac{1}{5}\ln|x + 1| + C.$$

99. Use integration by parts with $u = \sec^{-1}\sqrt{x}$, $dv = dx$.

100. Let $u = x^2$. The integral becomes

$$\frac{1}{2}\int \left(\frac{1 - u}{1 + u}\right)^{1/2} du.$$

Now let $v^2 = \frac{1 - u}{1 + u}$; then $u = \frac{1 - v^2}{1 + v^2}$ and

$$du = -\frac{4v}{(1 + v^2)^2}\,dv.$$

The integral is thereby converted into

$$-\frac{1}{2}\int \frac{4v^2}{(1 + v^2)^2}\,dv.$$

Finally let $v = \tan z$. Then we obtain

$$-2\int \sin^2 z\, dz = \sin z \cos z - z + C.$$

Subsequent resubstitutions and simplifications lead to the answer:

$$\frac{1}{2}(1 - x^4)^{1/2} - \tan^{-1}(\{1 - x^2\}/\{1 + x^2\})^{1/2} + C.$$

101. The area is

$$A = \int_0^1 2\pi \cosh^2 x\, dx$$

$$= 2\pi\left(\frac{1}{4}\sinh 2 + \frac{1}{2} - \frac{1}{4}\sinh 0 - 0\right)$$

$$= \frac{\pi}{4}\left(e^2 - \frac{1}{e^2} + 4\right) \sim 8.83865.$$

102. The length of the curve is

$$L = \int_0^1 (1 + e^{-2x})^{1/2}\, dx.$$

Now let $e^{-x} = \tan u$. Then

$$L = \int_{x=0}^1 -\frac{\sec^3 u}{\tan u}\, du$$

$$= -\int_{x=0}^1 (\csc u + \sec u \tan u)\, du$$

$$= \Big[-\ln|\csc u - \cot u| - \sec u\Big]_{x=0}^1$$