Chapter 9 Miscellaneous Problems

1. Let $x = u^2$.

2. $\ln|1 + \tan t| + C$

4. $-\tan^{-1}(\csc x) + C$, using $u = \csc x$ if you wish.

6. $\csc^4 x = (1 + \cot^2 x)(\csc^2 x) = \csc^2 x + \cot^2 x \csc^2 x$.

7. Let $u = x$, $dv = \tan^2 x\, dx = (\sec^2 x - 1)\, dx$; then $du = dx$ and $v = -x + \tan x$.

8. First write

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Integrate what you can, and then integrate $x^2 \cos 2x$ by parts: Let $u = x^2$ and $dv = \cos x\, dx$. This leads to the integration of $x \sin 2x$, which also works well by parts. You should get

$$\frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{8}\sin 2x + C.$$

9. Let $u = 2 - x^3$.

10. Let $x = 2\tan u$. The integrand becomes $\sec u$, and the antiderivative is

$$\ln|\sec u + \tan u| + C = \ln\left|\frac{1}{2}(x^2 + 4)^{1/2} + \frac{1}{2}x\right| + C_1$$

$$= \ln|x + (x^2 + 4)^{1/2}| + C.$$

11. Let $x = 5\tan \xi$. The integrand becomes $25(\sec^3 \xi - \sec \xi)$.

12. Let $u = \sin x$. The integrand becomes $(4 - u^2)^{1/2}$. Then let $u = 2\sin \xi$. The integrand then becomes $4\cos^2 \xi$, and the rest is easy. The answer may be written in the form

$$2\sin^{-1}\left(\frac{1}{2}\sin x\right) + \frac{1}{2}(\sin x)(4 - \sin^2 x)^{1/2} + C.$$

13. Write $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$, then apply Formula 17 from the endpapers.

14. First write $x^2 + x + 1$ as $\frac{3}{4}(u^2 + 1)$ where

$$u = \frac{2x + 1}{\sqrt{3}}.$$

Then let $u = \tan\xi$, and the integrand becomes $\frac{3}{4}\sec^3\xi$. Apply Formula 28 from the endpapers; the final answer is

$$\frac{1}{4}(2x + 1)(x^2 + x + 1)^{1/2}$$

$$+ \frac{3}{8}\ln|2x + 1 + 2(x^2 + x + 1)^{1/2}| + C.$$

15. $3x^2 - 4x + 11 = \frac{1}{3}\{(3x - 2)^2 + 29\}$.

16. The integrand is equal to

$$x^2 - 2 + \frac{5}{x^2 + 2}.$$

The antiderivative is

$$\frac{1}{3}x^3 - 2x + \frac{5}{2}\sqrt{2}\tan^{-1}(\frac{1}{2}x\sqrt{2}) + C.$$

17. Use the substitution $u = \tan\frac{\xi}{2}$. The integral becomes

$$\int \frac{2}{u^2 + 9}\,du = \frac{2}{3}\arctan(\frac{u}{3}) + C$$

$$= \frac{2}{3}\arctan(\frac{1}{3}\tan\frac{\xi}{2}) + C.$$

18. The substitution $x = u^2$ transforms the integrand into

$$2 - \frac{2}{1 + u^2}.$$

The answer is $2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$.

19. Use the substitution $u = \sin x$.

20. Write the numerator as $2\cos^2 x - 1$. The answer is

$$2\sin x - \ln|\sec x + \tan x| + C.$$

21. Let $u = \ln(\cos x)$, then compute du.

22. Let $x^2 = \sin u$. The integrand becomes $\frac{1}{2}\sin^3 u$, and the antiderivative is

$$\frac{1}{6}(\cos u)(\cos^2 u - 3) + C$$

$$= -\frac{1}{6}(x^4 + 2)(1 - x^4)^{1/2} + C.$$

23. By parts: Let $u = \ln(1 + x)$, $dv = dx$, and (!) $v = x + 1$.

24. By parts: Let $u = \sec^{-1}x$, $dv = x\,dx$. This leads to the problem of evaluating

$$\int \frac{|x|}{(x^2 - 1)^{1/2}}\,dx.$$

For $x > 1$, we obtain $\frac{1}{2}\{x^2\sec^{-1}x - (x^2 - 1)^{1/2}\} + C$.

For $x < -1$, we obtain $\frac{1}{2}\{x^2\sec^{-1}x + (x^2 - 1)^{1/2}\} + C$.

So for $|x| > 1$, the antiderivative is

$$\frac{1}{2}\{x^2\sec^{-1}x - \frac{|x|}{x}(x^2 - 1)^{1/2}\} + C.$$

25. Let $x = 3\tan u$, then parts with $dv = \sec^2\theta\,d\theta$

26. The substitution $x = 2\sin u$ yields

$$\int 4\sin^2 u\,du = 2\int(1 - \cos 2u)\,du$$

$$= 2u - 2\sin u \cos u + C$$

$$= 2\arcsin\frac{x}{2} - \frac{x}{2}(4 - x^2)^{1/2} + C.$$

27. $2x - x^2 = 1 - (1 - x)^2$. Now let $x = 1 - \cos u$.

28. $$\frac{4x - 2}{x^3 - x} = \frac{2}{x} - \frac{3}{x + 1} + \frac{1}{x - 1},$$

so the antiderivative is

$$2\ln|x| - 3\ln|x + 1| + \ln|x - 1| + C$$

$$= \ln\left|\frac{x^2(x - 1)}{(x + 1)^3}\right| + C.$$

29. Write the integrand in the form

$$x^2 + 2 - \frac{4}{2 - x^2}.$$

Then use Formula 18 from the endpapers, the method of partial fractions, or a trigonometric or hyperbolic substitution.

30. $$\int \frac{\sec x \tan x}{\sec x + \sec^2 x}\,dx = \int \frac{\tan x}{1 + \sec x}\,dx$$

$$= \int \frac{\sin x}{1 + \cos x}\,dx = -\ln(1 + \cos x) + C.$$

31. Let $x = -1 + \tan u$. The integral becomes

$$\int \frac{-1 + \tan u}{\sec^2 u}\,du = \int (-\cos^2 u + \sin u \cos u)\,du,$$

and the rest is routine.

32. The least common multiple of 2, 3, and 4 is 12, so let $u = x^{1/12}$. This transforms the integral into

$$\int \frac{12u^{12}}{u^3 + 1}\, du.$$

The result of division of the denominator of the integrand into the numerator yields

$$12(u^9 - u^6 + u^3 - 1 + \frac{1}{u^3 + 1}),$$

and the method of partial fractions yields the decomposition

$$\frac{12}{u^3 + 1} = \frac{4}{u + 1} - \frac{4u - 8}{u^2 - u + 1}.$$

Now proceed much as in the last part of the solution to Problem 8, Section 9.8, on pages 501 - 502. You'll find the antiderivative indicated at the top of this page to be

$$\frac{6}{5}u^{10} - \frac{12}{7}u^7 + 3u^4 - 12u + 4\ln|u + 1|$$

$$- 2\ln|u^2 - u + 1| + 4\sqrt{3}\arctan(\frac{1}{3}\sqrt{3}\{2u - 1\}) + C$$

$$= \frac{6}{5}x^{5/6} - \frac{12}{7}x^{7/12} + 3x^{1/3} - 12x^{1/12} + 4\ln(1 + x^{1/12})$$

$$- 2\ln(1 - x^{1/12} + x^{1/6}) + 4\sqrt{3}\arctan(\frac{1}{3}\sqrt{3}\{2x^{1/12} - 1\}) + C.$$

33. $\frac{1}{1 + \cos 2\xi} = \frac{1}{2}\sec^2\xi.$

34. $\int \frac{\sec x}{\tan x}\, dx = \int \csc x\, dx = \ln|\csc x - \cot x| + C.$

35. $\sec^3 x \tan^3 x = \sec^5 x \tan x - \sec^3 x \tan x.$