$\boxed{\text{Partial Fractions}}$

If $y = f(x)$ and $y = g(x)$ are polynomials, then it follows from a theorem in algebra that

$$\underbrace{\frac{f(x)}{g(x)}}_{\textbf{rational function}} \quad = \quad \underbrace{P(x)}_{\text{a polynomial}} \quad + \; \underbrace{F_1(x) \; + \; F_2(x) \; + \; \ldots \; + \; F_k(x)}_{\textbf{partial fraction decomposition}} \, , \tag{1}$$

where each **partial fraction** F_i has one of the forms

$$\frac{A}{(px+q)^m} \qquad \text{or} \qquad \frac{Cx+D}{(ax^2+bx+c)^n}$$

where

- m and n are nonnegative integers, i.e., $n, m \in \{0, 1, 2, 3, 4, 5, \ldots\}$
- $ax^2 + bx + c$ is irreducible, i.e., it cannot be factored over $\mathbb{R}$, i.e. $b^2 - 4ac < 0$.

Why do we care? Well, if (1) holds then

$$\underbrace{\int \frac{f(x)}{g(x)}\, dx}_{\textbf{we want to find this}} \quad = \quad \underbrace{\int P(x)\, dx}_{\text{easy to find}} \; + \; \underbrace{\int \left[F_1(x) \; + \; F_2(x) \; + \; \ldots \; + \; F_k(x)\right] dx}_{\textbf{do-able}} \; .$$

So how to find this decomposition ....

$$\boxed{\text{First Case:} \qquad [\text{degree of } y = f(x)] \; < \; [\text{degree of } y = g(x)]}$$

In this case, $P(x) = 0$ in (1). Express $y = g(x)$ as a product of

- linear factors $px + q$
- *irreducible* quadratic factors $ax^2 + bx + c$ (irreducible means that $b^2 - 4ac < 0$).

Collect up the repeated factors so that $g(x)$ is a product of *different* factors of the form $(px+q)^m$ and $(ax^2+bx+c)^n$. Then apply the following rules.

Rule 1: For each factor of the form $(px+q)^m$ where $m \geq 1$, the decomposition (1) contains a sum of m partial factions of the form

$$\frac{A_1}{(px+q)^1} \; + \; \frac{A_2}{(px+q)^2} \; + \; \ldots \; + \; + \; \frac{A_m}{(px+q)^m}$$

where each A_i is a real number.

Rule 2: For each factor of the form $(ax^2+bx+c)^n$ where $n \geq 1$ and $b^2 - 4ac < 0$, the decomposition (1) contains a sum of n partial factions of the form

$$\frac{A_1 x + B_1}{(ax^2+bx+c)^1} \; + \; \frac{A_2 x + B_2}{(ax^2+bx+c)^2} \; + \; \ldots \; + \; + \; \frac{A_n x + B_n}{(ax^2+bx+c)^n}$$

where the A_i's and B_i's are real number.

$$\boxed{\text{Second Case:} \qquad [\text{degree of } y = f(x)] \;\geq\; [\text{degree of } y = g(x)]}$$

First do long division to express $\frac{f(x)}{g(x)}$ as

$$\frac{f(x)}{g(x)} \;=\; \underbrace{P(x)}_{\text{a polynomial}} \;+\; \underbrace{\frac{R(x)}{g(x)}}_{[\text{degree of } y=R(x)] \;<\; [\text{degree of } y=g(x)]} \qquad ,$$

How to do this? Well we surely see that

$$\frac{5}{3} \;=\; 1\,\frac{2}{3} \;=\; 1 \;+\; \frac{2}{3}\;;$$

we get this by long division

$$\begin{array}{c} \quad 1 \\ 3\sqrt{5} \\ \\ 3 \\ \hline 2\;. \end{array}$$

Similarly,

$$\frac{f(x)}{g(x)} \;=\; P(x) \;+\; \frac{R(x)}{g(x)}\;,$$

where

$$\begin{array}{c} \quad \mathrm{P(x)} \\ g(x)\sqrt{f(x)} \\ \\ \vdots \\ \hline R(x)\;. \end{array}$$

Now we can apply the **First Case** to $\frac{R(x)}{g(x)}$ since $[\text{degree of } y = R(x)] \;<\; [\text{degree of } y = g(x)]$.

A common mistake. Note that $x^2 = (x-0)^2 = 1x^2 + 0x + 0$ and so $b^2 - 4ac = 0 \not< 0$. So we follow **Rule 1** to see that the partial fraction decomposition of $\frac{1}{x^2}$ is of the form

$$\frac{1}{x^2} \;=\; \frac{A}{x^1} \;+\; \frac{B}{x^2}\;.$$

Note that $A = 0$ and $B = 1$. A common mistake is to try to use **Rule 2**, which would give

$$\frac{1}{x^2} \quad \overset{\text{wrong}}{=} \quad \frac{Ex+F}{x^1} \;+\; \frac{Gx+H}{x^2}\;.$$

This would still lead to the correct answer ($E = F = G = 0$ and $H = 1$) but you have to do LOTS of work to get to it.