

- The book gives lots of *rules/strategies*.
Read them but do NOT memorize.
Use logic instead, as done below!
- Recall the following natural pairings (in terms of derivatives):

$$\sin \leftrightarrow \cos \quad \tan \leftrightarrow \sec \quad \cot \leftrightarrow \csc$$

If the integrand does not contain natural pairings, then rewrite it so that it does.

Eg: $\tan x \cos^2 x = \frac{\sin x}{\cos x} \cos^2 x = \sin x \cos x$.

- Make a natural choice for s or t . For example, if the integrand involves sine and cosine, then try

$s = \cos x$
$ds = -\sin x dx$

or
$t = \sin x$
$dt = \cos x dx$

If the integrand involves tangent and secant, then try $s = \tan x$ or $t = \sec x$.

If the integrand involves cotangent and cosecant, then try $s = \cot x$ or $t = \csc x$.

- Then isolate off your ds or dt and try to express what is left in terms of just s or t .
- The following will be helpful (and you must memorize them):

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos(2x)}{2} \quad (\text{half-angle formulas})$$

$$\cos^2 x + \sin^2 x = 1 \implies \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \implies 1 + \tan^2 x = \sec^2 x$$

$$\cos^2 x + \sin^2 x = 1 \implies \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \implies \cot^2 x + 1 = \csc^2 x$$

- For integrals of the form $\int \sin mx \cos nx dx$ or $\int \sin mx \sin nx dx$ or $\int \cos mx \cos nx dx$

where $m \neq n$, use the trig identities (these 3 you do not have to memorize)

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)] \quad (1)$$

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \quad (2)$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] . \quad (3)$$

Example 4. $\int \sin^4 x dx$

If we try $s = \cos x$ or $t = \sin x$, it will not work (why? $\int \sin^4 x dx = - \int [\sin^3 x] [-\sin x dx]$).
Here we use the half-angle formulas.

$$\begin{aligned} \int \sin^4 x dx &= \int [\sin^2 x]^2 dx = \int \left[\frac{1 - \cos(2x)}{2} \right]^2 dx = \frac{1}{4} \int [1 - 2\cos(2x) + \cos^2(2x)] dx \\ &= \frac{1}{4} \int \left[1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2} \right] dx \\ &= \frac{3}{8} \int dx - \frac{1}{4} \cdot \int \cos(2x) 2dx + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \int \cos(4x) 4dx \\ &= \frac{3}{8}x - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C \end{aligned}$$

Example 1. $\int \sin^7 x dx$

$$\begin{cases} s = \cos x \\ ds = -\sin x dx \end{cases}$$

or

$$\begin{cases} t = \sin x \\ dt = \cos x dx \end{cases}$$

s-sub $\int \sin^7 x dx = - \int [\sin^6 x] [-\sin x dx]$

Now can we express $[\sin^6 x]$ in terms of only $s = \cos x$? Yes, easily!

$$\sin^6 x = (\sin x)^6 = (\sin^2 x)^3 = (1 - \cos^2 x)^3$$

So:

$$\begin{aligned} \int \sin^7 x dx &= - \int [\sin^6 x] [-\sin x dx] \\ &= - \int [1 - \cos^2 x]^3 [-\sin x dx] \\ &= - \int (1 - s^2)^3 ds \\ &= \int (s^6 - 3s^4 + 3s^2 - 1) ds \\ &= \frac{s^7}{7} - \frac{3s^5}{5} + \frac{3s^3}{3} - s + C \\ &= \frac{\cos^7 x}{7} - \frac{3\cos^5 x}{5} + \cos^3 x - \cos x + C \end{aligned}$$

t-sub $\int \sin^7 x dx = \int [\text{???}] [\text{would want here } \cos x dx]$

The substitution with $t = \sin x$ doesn't work because there is no $\cos x$ in the original integrand.

Example 2. $\int \sin^4(17x) \cos^3(17x) dx$

$$\boxed{s = \cos(17x)} \\ \boxed{ds = -17 \sin(17x) dx}$$

or

$$\boxed{t = \sin(17x)} \\ \boxed{dt = 17 \cos(17x) dx}$$

s-sub $\int \sin^4(17x) \cos^3(17x) dx = \frac{-1}{17} \int [\cos^3(17x) \sin^3(17x)] [-17 \sin(17x) dx].$

Can we express $\boxed{\cos^3(17x) \sin^3(17x)}$ in terms of only $s = \cos(17x)$?

Not easily, there is an “extra sin”:

$$\cos^3(17x) \sin^3(17x) = \cos^3(17x) \sin^2(17x) \sin(17x) = \cos^3(17x) [1 - \cos^2(17x)] \sin(17x)$$

t-sub $\int \sin^4(17x) \cos^3(17x) dx = \frac{1}{17} \int [\sin^4(17x) \cos^2(17x)] [17 \cos(17x) dx].$

Can we express $\boxed{\sin^4(17x) \cos^2(17x)}$ in terms of only $t = \sin(17x)$? Yes.

$$\sin^4(17x) \cos^2(17x) = \sin^4(17x) (1 - \sin^2(17x)) .$$

So:

$$\begin{aligned} \int \sin^4(17x) \cos^3(17x) dx &= \frac{1}{17} \int [\sin^4(17x) \cos^2(17x)] [17 \cos(17x) dx] \\ &= \frac{1}{17} \int [\sin^4(17x) (1 - \sin^2(17x))] [17 \cos(17x) dx] \\ &= \frac{1}{17} \int t^4 (1 - t^2) dt \\ &= \frac{1}{17} \int (t^4 - t^6) dt \\ &= \frac{1}{17} \left(\frac{t^5}{5} - \frac{t^7}{7} \right) + C \\ &= \frac{1}{17} \left(\frac{\sin^5(17x)}{5} - \frac{\sin^7(17x)}{7} \right) + C \end{aligned}$$

Example 3. $\int \tan \theta \sec^4 \theta d\theta$

$$\begin{aligned} s &= \tan \theta \\ ds &= \sec^2 \theta d\theta \end{aligned}$$

or

$$\begin{aligned} t &= \sec \theta \\ dt &= \sec \theta \tan \theta d\theta \end{aligned}$$

s-sub $\int \tan \theta \sec^4 \theta d\theta = \int [\tan \theta \sec^2 \theta] [\sec^2 \theta d\theta]$

Can we express $\tan \theta \sec^2 \theta$ in terms of only $s = \tan \theta$? Yes.

$$\tan \theta \sec^2 \theta = \tan \theta (1 + \tan^2 \theta)$$

So:

$$\begin{aligned} \int \tan \theta \sec^4 \theta d\theta &= \int [\tan \theta \sec^2 \theta] [\sec^2 \theta d\theta] = \int [\tan \theta (1 + \tan^2 \theta)] [\sec^2 \theta d\theta] \\ &= \int s(1 + s^2) ds = \int (s + s^3) ds \\ &= \frac{s^2}{2} + \frac{s^4}{4} + C \\ &= \frac{\tan^2 \theta}{2} + \frac{\tan^4 \theta}{4} + C \end{aligned}$$

t-sub $\int \tan \theta \sec^4 \theta d\theta = \int [\sec^3 \theta] [\sec \theta \tan \theta d\theta]$

Can we express $\sec^3 \theta$ in terms of only $t = \sec \theta$? Of course. So:

$$\begin{aligned} \int \tan \theta \sec^4 \theta d\theta &= \int [\sec^3 \theta] [\sec \theta \tan \theta d\theta] \\ &= \int t^3 dt = \frac{t^4}{4} + k \\ &= \frac{\sec^4 \theta}{4} + k \end{aligned}$$

Why are these answers different? They're not.

$$\begin{aligned} \frac{\sec^4 \theta}{4} + k &= \frac{1}{4} (\sec^2 \theta)^2 + k = \frac{1}{4} (1 + \tan^2 \theta)^2 + k \\ &= \frac{1}{4} (1 + 2 \tan^2 \theta + \tan^4 \theta) + k = \frac{\tan^2 \theta}{2} + \frac{\tan^4 \theta}{4} + \left(k + \frac{1}{4}\right) \end{aligned}$$

So $C = k + \frac{1}{4}$.