

Tue 1:00

NAME: Key**INSTRUCTIONS:**

1. To receive credit you must:
  - a. WORK IN A LOGICAL FASHION,  
SHOW ALL YOUR WORK,  
INDICATE YOUR REASONING.
  - b. when applicable put your answer on/in the line/box provided
  - c. if no such line/box is provided, then box your answer
  - d. if you use your calculator, give an explanation of what you did on it.
2. There are 18 pages to this exam: a total of 15 equally weighted problems. Check that your copy of the exam has all of the pages.
3. During this test, do not leave your seat.  
If you have a question, raise your hand.  
When you finish: turn your exam over, put your pencil down, raise your hand.
4. This closed book/notes exam covers (from *Calculus*, by Varberg and Purcell) :  
§ 7.1 - 7.7 , 8.1 - 8.6 , 9.1 - 9.4 , 10.1 - 10.3 , 11.1 - 11.8 , 12.6 - 12.8 .

**Problem Source:**

1. Fall 1998 Exam 1 # 1c
  2. Fall 1998 Exam 1 # 3, simplified
  3. Fall 1998 Exam 2 # 1a
  4. Fall 1998 Exam 2 # 1c
  5. Fall 1998 Exam 2 # 2a, simplified
  6. Fall 1998 Exam 2 # 4c
  7. Fall 1998 Exam 3 # 3d
  8. Fall 1998 Exam 3 # 3b
  9. Fall 1998 Exam 4 # 1a
  10. Example from class
  11. Similar to homework problems § 11.7 # 1-8 and examples from class
  12. Similar to examples from class
  13. Fall 1998 Exam 2 # 5
  14. Fall 1998 Exam 3 # 4
  15. An example from class: § 8.4 # 85
- bonus. the quote of our class!

BONUS:

Any fool can know , the point is to understand .  
Albert Einstein

$$1. \frac{d}{dx} e^{\tan x} = (\sec^2 x) e^{\tan x}$$

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$$2. \frac{d}{dx} (\ln x)^{2x+3} = \underline{(\ln x)^{2x+3} \left[ 2 \ln(\ln x) + \frac{2x+3}{x \ln x} \right]}$$

HINT: of the form  $\frac{d}{dx} ([f(x)]^{g(x)})$ .

$$y = (\ln x)^{2x+3}$$

$$\ln y = (2x+3) \ln(\ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = 2 \ln(\ln x) + (2x+3) \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$3. \int \sin^3(2x) dx = \underline{-\frac{1}{2} \cos 2x + \frac{1}{6} \cos^3 2x} + C.$$

$$= \int (1 - \cos^2 2x) \sin 2x dx$$

$$u = \cos 2x$$

$$du = -2 \sin 2x dx$$

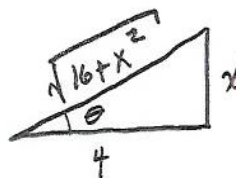
$$= -\frac{1}{2} \int (1 - u^2) du$$

$$= -\frac{1}{2} \left[ u - \frac{u^3}{3} \right] + C$$

$$4. \int \frac{dx}{(16+x^2)^{3/2}} = \frac{x}{16(16+x^2)^{1/2}} + C.$$

Your answer should NOT contain inverse trig functions ... convert back.

$$\begin{aligned} x &= 4 \tan \theta \\ dx &= 4 \sec^2 \theta d\theta \\ 16+x^2 &= 16 \sec^2 \theta \end{aligned}$$



$$= \int \frac{4 \sec^2 \theta d\theta}{(4 \sec \theta)^{2 \cdot 3/2}} = \int \frac{4 \sec^2 \theta d\theta}{(4 \sec \theta)^3}$$

$$= \frac{1}{4^2} \int \cos \theta d\theta = \frac{1}{16} \sin \theta + C$$

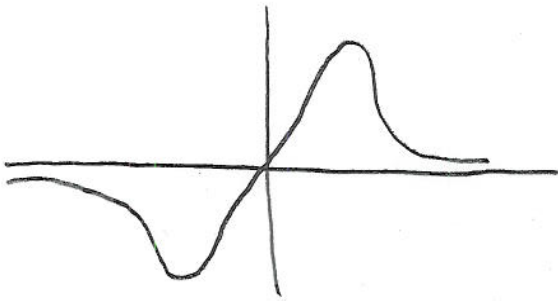
$$5. \int \ln x dx = \underline{x \ln x - x} + C.$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\int \ln x dx = x \ln x - \int dx$$

$$6. \int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \text{DNE}$$



$$\int_0^b \frac{x dx}{x^2+1} = \frac{1}{2} \ln|x^2+1| \Big|_0^b = \frac{1}{2} \ln|b^2+1| \rightarrow \infty$$

$$\int_a^0 \frac{x dx}{x^2+1} = \frac{1}{2} \ln|x^2+1| \Big|_a^0 = -\frac{1}{2} |a^2+1| \rightarrow -\infty$$

$$7. \lim_{n \rightarrow \infty} (-1)^n \frac{100n}{n^{3/2} + 4} = \underline{0}$$

8.  $\lim_{n \rightarrow \infty} (-1)^n \frac{6n - 5}{5n + 1} = \underline{\text{DNE}}$

9.  $\sum_{n=7}^{\infty} (-1)^n \frac{n}{1+n^2}$

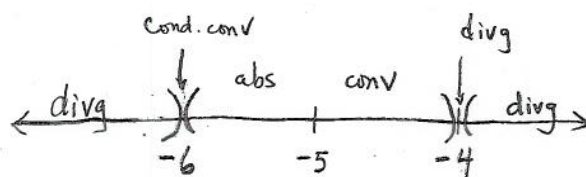
- absolutely convergent  
 conditionally convergent  
 divergent

10. Indicate precisely where the series

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n}$$

is absolutely convergent, conditionally convergent, and divergent (a diagram as we did in class suffices).

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(x+5)^{n+1}}{n+1} \cdot \frac{n}{(x+5)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot |x+5| = |x+5| < 1$$



$$\sum \frac{(x+5)^n}{n} \quad \underline{x = -6} \quad \sum \frac{(-1)^n}{n} \quad \leftarrow \text{cond. conv.}$$

$$\sum \frac{(x+5)^n}{n} \quad \underline{x = -4} \quad \sum \frac{1}{n} \quad \leftarrow \text{divg}$$

11. Using the Geometric Series, find a power series representation for

$$f(x) = \frac{7}{9 + 4x^2}$$

and specify precisely when (i.e., for what values of  $x$ 's?) this representation is valid.

$$\frac{7}{9 + 4x^2} = \frac{7}{9} \cdot \frac{1}{1 + \frac{4x^2}{9}}$$

$$= \frac{7}{9} \sum_{n=0}^{\infty} \left(-\frac{4x^2}{9}\right)^n$$

valid :  $\left| \frac{4x^2}{9} \right| < 1$

$$|x|^2 < \frac{9}{4}$$

$$\boxed{|x| < \frac{3}{2}}$$

12. Grandpa and Grandma Puffo are out for their morning walk over a time period of  $0 \leq \theta \leq 2\pi$ . Grandpa is walking along a path given by  $r = \sin \theta$  while Grandma is walking along a path given by  $r = \sqrt{3 \cos^2 \theta}$ .
- 12a. Fill in the below chart, similar to as we did in class.

| $\theta$                  | $r = \sin \theta$      | $r = \sqrt{3 \cos^2 \theta}$ |
|---------------------------|------------------------|------------------------------|
| $0 \rightarrow \pi/2$     | $0 \xrightarrow{1} 1$  | $\sqrt{3} \xrightarrow{5} 0$ |
| $\pi/2 \rightarrow \pi$   | $1 \xrightarrow{2} 0$  | $0 \xrightarrow{6} \sqrt{3}$ |
| $\pi \rightarrow 3\pi/2$  | $0 \xrightarrow{3} -1$ | $\sqrt{3} \xrightarrow{7} 0$ |
| $3\pi/2 \rightarrow 2\pi$ | $-1 \xrightarrow{4} 0$ | $0 \xrightarrow{8} \sqrt{3}$ |

- 12b. Sketch below the curves of Grandpa and Grandma's walking paths, indicating parts 1-8, as we did in class

- 12c. GIVE POINTS IN POLAR FORM:  $(r, \theta)$ .

Grandpa and Grandma Puffo will pass each other, and be able to give each other

HIGH-FIVES, at time/times  $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

At this time/these times, they will be at the point/points  $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$  &  $(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$

Grandpa and Grandma Puffo will pass through the same place, but not at the same time (so they will not be able to give each other HIGH-FIVES), at point/points

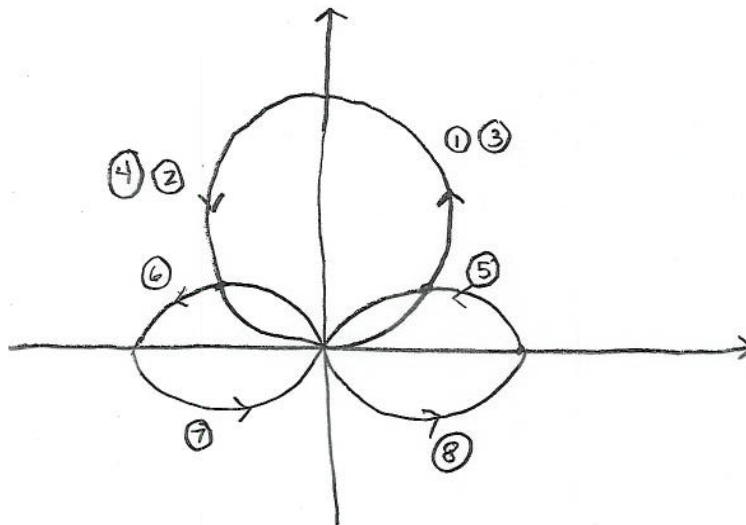
origin

$$\sin \theta = \sqrt{3 \cos^2 \theta}$$

$$\sin^2 \theta = 3 \cos^2 \theta$$

$$\tan \theta = \sqrt{3} = \frac{\sqrt{3}/2}{1/2}$$

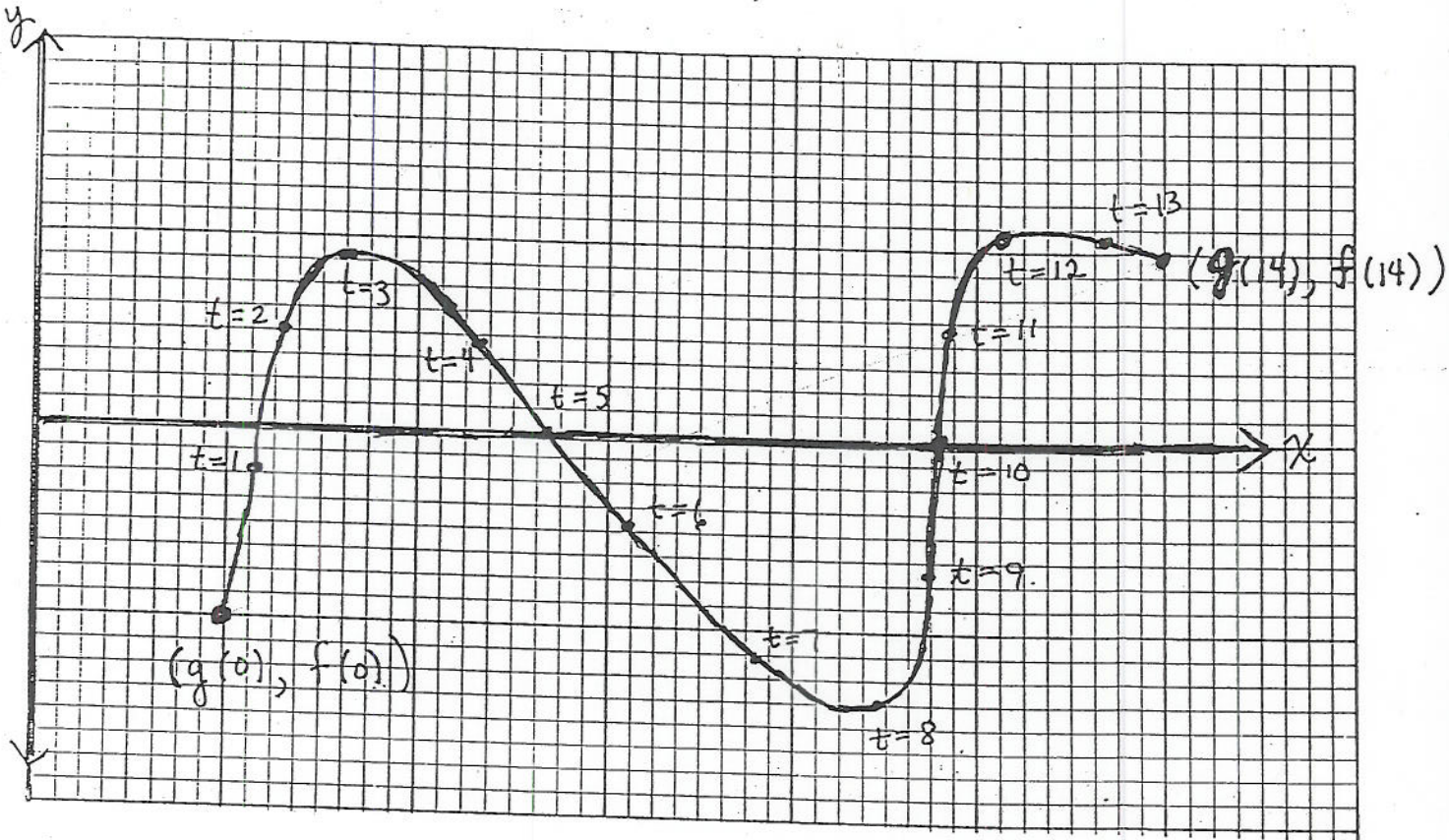
$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$



13. Let  $f: [0, 14] \rightarrow \mathbb{R}$  and  $g: [0, 14] \rightarrow \mathbb{R}$  be two continuous functions, each differentiable on  $(0, 14)$ . Also  $g'(x) \neq 0$  for each  $x \in (0, 14)$ . Cauchy Puffetto is moving along the curve  $\mathcal{C}$  show below in such a way that his position at time  $t$ , where  $0 \leq t \leq 14$ , is  $(x, y)$  where  $x = g(t)$  and  $y = f(t)$ . Estimate all points  $c$ , where  $c \in (0, 14)$ , for which

$$\frac{f(14) - f(0)}{g(14) - g(0)} = \frac{f'(c)}{g'(c)} = \frac{df/dt}{dg/dt} = \frac{dy}{dx}$$

ESTIMATE(S) ON THE C('S): 3, 8, 12  
 Clearly geometrically explain your reasoning! One point for the correct estimate(s) and 4 points for your explanation.



$$\frac{f(b) - f(a)}{b - a} = \frac{f'(c)}{1}$$

more space provided  $\rightarrow$

More space for # 13:

14. Formally show (i.e., by using the definition of convergence: page 514) that

$$\lim_{n \rightarrow \infty} \frac{2n^3}{n^3 + 1} = 2.$$

Hints: I got you started, just fill in the missing parts. You might find helpful the observation that  $\frac{2}{n^3+1} < \frac{2}{n^3}$ .

*Proof.*

Recall that by definition of convergence of a sequence,

$$\lim_{n \rightarrow \infty} \frac{2n^3}{n^3 + 1} = 2$$

if and only if for each  $\varepsilon > 0$  there exists a number  $N_\varepsilon > 0$  such that if  $n \geq N_\varepsilon$  then

$$\left| \frac{2n^3}{n^3 + 1} - 2 \right| < \varepsilon.$$

Let an arbitrary  $\varepsilon > 0$  be given. Let  $N_\varepsilon = \underline{\hspace{2cm}}$ . Then  $n \geq N_\varepsilon$  implies that

$$\left| \frac{2n^3}{n^3 + 1} - 2 \right| = \left| \frac{2n^3 - 2(n^3 + 1)}{n^3 + 1} \right| = \frac{2}{n^3 + 1}$$

$$\frac{2}{n^3 + 1} < \frac{2}{n^3} \leq \frac{2}{N_\varepsilon^3} = \varepsilon \quad \Leftrightarrow \quad \sqrt[3]{\frac{2}{\varepsilon}} \equiv N_\varepsilon$$

(or)

$$\frac{2}{n^3 + 1} \leq \frac{2}{N_\varepsilon^3 + 1} = \varepsilon \quad \Leftrightarrow \quad \sqrt[3]{\frac{2}{\varepsilon} - 1} \equiv N_\varepsilon$$

Thus, by the definition of convergence of a sequence,

$$\lim_{n \rightarrow \infty} \frac{2n^3}{n^3 + 1} = 2. \quad \blacksquare$$

15. Let  $\mathbb{N}$  be the set of natural numbers, i.e.

$$\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}.$$

Consider a function

$$f: [-\pi, \pi] \rightarrow \mathbb{R}$$

whose derivative

$$f': [-\pi, \pi] \rightarrow \mathbb{R}$$

is continuous. For  $n \in \mathbb{N}$ , defined the  $n^{\text{th}}$  Fourier coefficient of  $f$  to be

$$a_n \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

$$u = f(x)$$

$$dv = \sin(nx) dx$$

$$du = f'(x) dx$$

$$v = -\frac{1}{n} \cos nx$$

Show that  $\lim_{n \rightarrow \infty} a_n = 0$ . Hint: integration by parts.

$$a_n = \frac{1}{\pi} \left[ \left. -\frac{f(x) \cos nx}{n} \right|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \cos nx dx \right]$$

$$= \frac{-f(\pi) \cos n\pi}{\pi n} + \frac{f(-\pi) \cos n\pi}{\pi n} + \frac{\pi}{n} \int_{-\pi}^{\pi} f'(x) \cos nx dx$$

more space provided  $\longrightarrow$

More space for # 15: