

MARK BOX		
PROBLEM	POINTS	
1 a-e	25	
2 a-c	15	
3 a-f	30	
4	5	
Total	75	
%	100	

Math 142.501

Prof. Girardi

Fall 98

Exam 3

11/17/98

NAME: Key

### INSTRUCTIONS:

- To receive credit you must:
  - WORK IN A LOGICAL FASHION,  
SHOW ALL YOUR WORK,  
INDICATE YOUR REASONING.
  - when applicable put your answer on/in the line/box provided
  - if no such line/box is provided, then box your answer
  - if you use your calculator, give an explanation of what you did on it.
- The MARK BOX indicates the problems along with their points.  
Check that your copy of the exam has all of the problems.
- Do not discuss this exam with anyone other than yourself.
- This **OPEN** book/notes test covers (from *Calculus*, by Varberg and Purcell) :  
§ 10.1 – 10.3, 11.1 .

### Problem Source:

- Inspired by look at problem § 10.6, Sample Test Problem # 8.
- Similar to a Puffo problem from class.
- Taken from my high-school calculus textbook.
- Similar to an example from class and Example 1 from § 11.1.

BONUS: The first line of *The Carolinian Creed* is:

I will practice personal and academic integrity.

1. Numerical Integration with the function  $f(x) = \ln x$ . Let's approximate  $\int_{0.8}^{1.2} \ln x dx$ .

1a. Fill in the below chart.

$n$	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}$
0	$\ln(x)$	0	0
1	$x^{-1}$	1	1
2	$-x^{-2}$	-1	$-\frac{1}{2}$
3	$2x^{-3}$	2!	$\frac{2!}{3!} = \frac{1}{3}$
4	$-3!x^{-4}$	-3!	$-\frac{3!}{4!} = -\frac{1}{4}$
5	$4!x^{-5}$	4!	$\frac{4!}{5!} = \frac{1}{5}$

ps. my TI-83 says  $\int_{0.8}^{1.2} \ln x \approx -0.0026993$

1b. From the pattern in the chart, I see that for  $n \geq 1$ ,

$$f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}$$

and

$$\frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n-1}}{n}$$

Hint: simplify the above as much as possible, it will make your life easier later on.

1c. For  $n \geq 1$ , the Taylor polynomial  $P_n$  of order  $n$  about  $a = 1$  for  $f(x) = \ln x$  is:

$$P_n(x) = \underline{1} (x-1)^1 + \underline{-\frac{1}{2}} (x-1)^2 + \underline{\frac{1}{3}} (x-1)^3 + \dots + \underline{\frac{(-1)^{n-1}}{n}} (x-1)^n$$

and the order  $n$  Remainder Term is:

$$\ln x - P_n(x) \stackrel{\text{def}}{=} R_n(x) = \frac{(-1)^n (x-1)^{n+1}}{(n+1) c^{n+1}}$$

where  $c$  is between  $\underline{x}$  and  $\underline{1}$ .

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

$$= \frac{(-1)^n n!}{c^{n+1}} \cdot \frac{1}{(n+1)!} (x-1)^{n+1}$$

1d. From 1c, we see that

$$\int_{0.8}^{1.2} \ln x \, dx \approx \int_{0.8}^{1.2} P_n(x) \, dx \quad (1d.1)$$

where an upper bound on the error of approximation is:

$$\left| \int_{0.8}^{1.2} R_n(x) \, dx \right|. \quad (1d.2)$$

Find  $n$  (as small as you can get it) so that the formula in 1d.2 insures that the approximation in 1d.1 is accurate to 4 decimal places.

Hint:  $\int_{0.8}^{1.2} |x-1|^{n+1} \, dx = 2 \int_1^{1.2} (x-1)^{n+1} \, dx$ .

$$\left| \int_{0.8}^{1.2} R_n(x) \, dx \right|$$

$$\leq \int_{0.8}^{1.2} |R_n(x)| \, dx$$

$$= \frac{1}{n+1} \int_{0.8}^{1.2} \frac{|x-1|^{n+1}}{c^{n+1}} \, dx$$

$$\leq \frac{1}{n+1} \int_{0.8}^{1.2} \frac{|x-1|^{n+1}}{(0.8)^{n+1}} \, dx$$

$$= \frac{2}{(n+1)(0.8)^{n+1}} \int_1^{1.2} (x-1)^{n+1} \, dx$$

$$= \frac{2}{(n+1)(0.8)^{n+1}} \frac{(x-1)^{n+2}}{n+2} \Big|_1^{1.2}$$

$$= \frac{2}{(n+1)(n+2)} \frac{(0.2)^{n+2}}{(0.8)^{n+1}} = \frac{2}{(n+1)(n+2)} \frac{(0.2)}{4^{n+1}} = \frac{2}{(n+1)(n+2) 5 \cdot 4 \cdot 4^n}$$

$$= \frac{1}{10(n+1)(n+2) 4^n} \stackrel{\text{WANT}}{\leq} \frac{.00005}{4}$$

$n$	$\frac{1}{10(n+1)(n+2)4^n}$
3	$\frac{1}{10 \cdot 4 \cdot 5 \cdot 4^3} \approx .000078$
4	$\frac{1}{10 \cdot 5 \cdot 6 \cdot 4^4} \approx .0000132$ 60208

$n = 4$

hp.

1e. Using a Taylor polynomial, approximate  $\int_{0.8}^{1.2} \ln x dx$  to 4 decimal places.

$$\int_{0.8}^{1.2} \ln x dx \approx \underline{\quad - .0026 \quad}$$

$$\int_{0.8}^{1.2} \ln x dx \stackrel{4dp}{\approx} \int_{0.8}^{1.2} P_4(x) dx$$

$$= \int_{0.8}^{1.2} \left[ (x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \right] dx$$

$$= \left. \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{20} \right|_{0.8}^{1.2}$$

$$= \left[ \frac{(.2)^2}{2} - \frac{(.2)^3}{6} + \frac{(.2)^4}{12} - \frac{(.2)^5}{20} \right]$$

$$- \left[ \frac{(.2)^2}{2} - \frac{-.2^3}{6} + \frac{(.2)^4}{12} - \frac{-.2^5}{20} \right]$$

$$= -2 \left[ \frac{(.2)^3}{6} + \frac{(.2)^5}{20} \right]$$

$$= -2 \left[ \frac{1}{5^3 \cdot 6} + \frac{1}{5^5 \cdot 20} \right]$$

$$= -\frac{2}{2 \cdot 5^3} \left[ \frac{1}{3} + \frac{1}{250} \right] = -\frac{1}{5^3} \left[ \frac{250+3}{750} \right]$$

$$= -\frac{253}{5^3 \cdot 5^3 \cdot 6} = -\frac{253}{5^6 \cdot 6} \approx \underline{\underline{-0026,986667}}$$

↑  
take this much.

bc that (just for fun)

$$-0026986667 - .0000130208$$

$$-0027116875$$

$$\int_{0.8}^{1.2} \ln x dx \stackrel{5 || TI-83}{\leq} -0026986667 + .0000130208$$

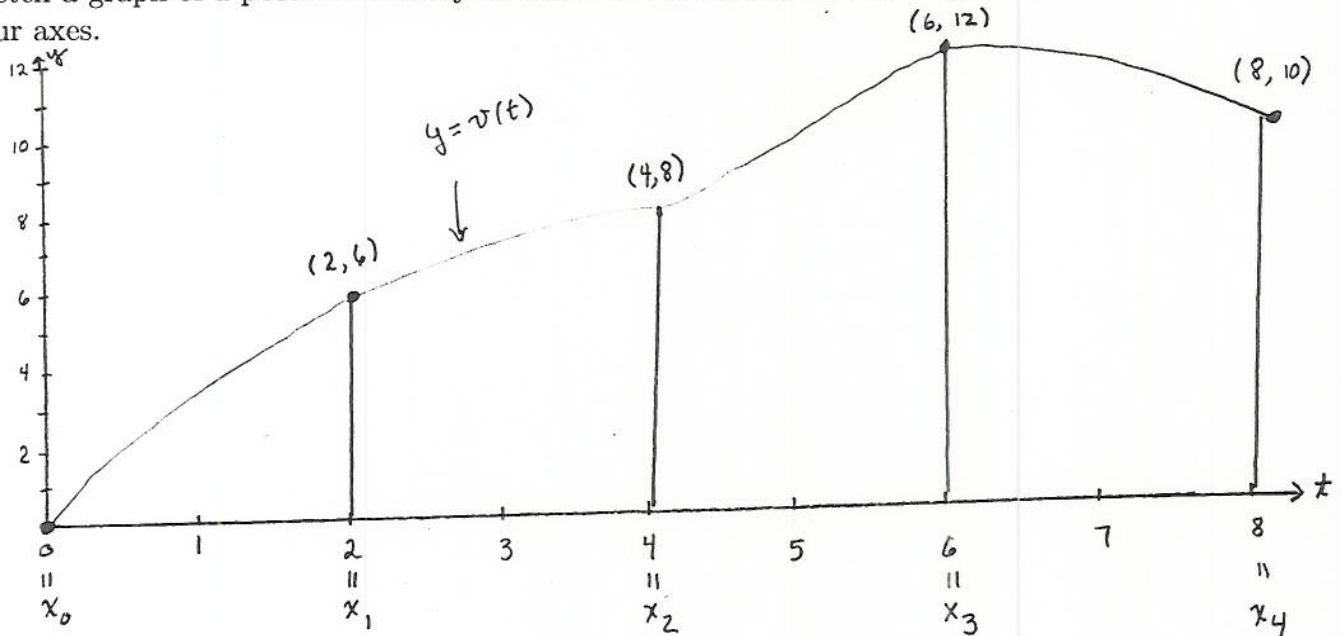
$$\leq -0026993 \leq -0026856459$$

2. Mr. Puffo is moving along the real number line. He observes his velocity at some key times as:

$t$	0	2	4	6	8
$v(t)$	0	6	8	12	10

The units are minutes and feet.

- 2a. Sketch a graph of a possible velocity function for Mr. Puffo. Do not forget to label your axes.



$$\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$$

$$\text{distance} = \int v(t) dt$$

2b. The Trapezoidal Rule (with  $n = 4$ ) gives that Mr. Puffo travel approximately

62 feet in these 8 minutes.

$$\begin{aligned}T_4 &= \frac{\Delta x}{2} [v(x_0) + 2v(x_1) + 2v(x_2) + 2v(x_3) + 1v(x_4)] \\&= \frac{2}{2} [1 \cdot v(0) + 2v(2) + 2v(4) + 2v(6) + 1 \cdot v(8)] \\&= 1 [1 \cdot 0 + 2 \cdot 6 + 2 \cdot 8 + 2 \cdot 12 + 1 \cdot 10] \\&= 0 + 12 + 16 + 24 + 10 \\&= 62\end{aligned}$$

2c. The Parabolic Rule (with  $n = 4$ ) gives that Mr. Puffo travel approximately

$65 \frac{1}{3}$  feet in these 8 minutes.

$$\begin{aligned}P_4 &= \frac{\Delta x}{3} [1 \cdot v(x_0) + 4v(x_1) + 2v(x_2) + 4v(x_3) + 1v(x_4)] \\&= \frac{2}{3} [1 \cdot v(0) + 4v(2) + 2v(4) + 4v(6) + 1v(8)] \\&= \frac{2}{3} [1 \cdot 0 + 4 \cdot 6 + 2 \cdot 8 + 4 \cdot 12 + 1 \cdot 10] \\&= \frac{2}{3} [24 + 16 + 48 + 10] \\&= \frac{2 \cdot 98}{3} = \frac{196}{3} = 65 \frac{1}{3}\end{aligned}$$

3. Determine if the limits of the following sequences exist. If your answer is DNE, explain why.

3a.  $\lim_{n \rightarrow \infty} \frac{6n-5}{5n+1} = \underline{\frac{6}{5}}$

$$\lim_{x \rightarrow \infty} \frac{6x-5}{5x+1} \stackrel{\text{Algebra}}{=} \lim_{x \rightarrow \infty} \frac{6 - \frac{5}{x}}{5 + \frac{1}{x}} = \frac{6}{5}$$

3b.  $\lim_{n \rightarrow \infty} (-1)^n \frac{6n-5}{5n+1} = \underline{\text{DNE} - \text{oscillating btw. } \approx \frac{6}{5} \text{ \& } \approx -\frac{6}{5}}$

$$3c. \lim_{n \rightarrow \infty} \frac{100n}{n^{3/2} + 4} = \underline{0}$$

$$\lim_{x \rightarrow \infty} \frac{100x}{x^{3/2} + 4} = \lim_{x \rightarrow \infty} \frac{\frac{100}{x^{1/2}}}{1 + \frac{4}{x^{3/2}}} = \frac{0}{1+0}$$

$$3d. \lim_{n \rightarrow \infty} (-1)^n \frac{100n}{n^{3/2} + 4} = \underline{0}$$

$$\begin{array}{ccc} -\frac{100n}{n^{3/2} + 4} & \leq & (-1)^n \frac{100n}{n^{3/2} + 4} \leq \frac{100n}{n^{3/2} + 4} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Squeeze Theorem.

$$3e. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = \underline{0}$$

$$\begin{array}{ccc} -\frac{1}{n} & \leq & \frac{\sin n}{n} \leq \frac{1}{n} \\ \downarrow n \rightarrow \infty & & \downarrow n \rightarrow \infty \\ 0 & & 0 \end{array}$$

Squeeze Theorem

$$3f. \lim_{n \rightarrow \infty} \sin(n\pi) = \underline{0}$$

note  $\sin(n\pi) = 0 \quad \forall n \in \mathbb{N}$ .

4. Formally show (i.e., by using the definition of convergence: page 514) that

$$\lim_{n \rightarrow \infty} \frac{2n^3}{n^3 + 1} = 2.$$

Hints: I got you started, just fill in the missing parts. You might find helpful the observation that  $\frac{2}{n^3+1} < \frac{2}{n^3}$ .

*Proof.*

Recall that by definition of convergence of a sequence,

$$\lim_{n \rightarrow \infty} \frac{2n^3}{n^3 + 1} = 2$$

if and only if for each  $\varepsilon > 0$  there exists a number  $N_\varepsilon > 0$  such that if  $n \geq N_\varepsilon$  then

$$\left| \frac{2n^3}{n^3 + 1} - 2 \right| < \varepsilon.$$

Let an arbitrary  $\varepsilon > 0$  be given. Let  $N_\varepsilon = \sqrt[3]{\frac{2}{\varepsilon}}$ . Then  $n \geq N_\varepsilon$  implies that

$$\left| \frac{2n^3}{n^3 + 1} - 2 \right| = \left| \frac{2n^3 - 2(n^3 + 1)}{n^3 + 1} \right|$$

$$= \left| \frac{-2}{n^3 + 1} \right|$$

$$= \frac{2}{n^3 + 1}$$

$$< \frac{2}{n^3} < \frac{2}{N_\varepsilon^3} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon.$$

$\uparrow$   
 $N_\varepsilon \leq n$

Thus, by the definition of convergence of a sequence,

$$\lim_{n \rightarrow \infty} \frac{2n^3}{n^3 + 1} = 2. \quad \blacksquare$$