MARK BOX		
PROBLEM	POINTS	
1 a–d	20	
2 a–d	20	
3	10	
4	10	
5	10	
6	10	
Total	80	
%	100	

Math 142.501 Prof. Girardi

Fall 98 Exam 1 9/17/98

NAME: Library

INSTRUCTIONS:

1. To receive credit you must:
   a. WORK IN A LOGICAL FASHION,
      SHOW ALL YOUR WORK,
      INDICATE YOUR REASONING.
   b. when applicable put your answer on/in the line/box provided
   c. if no such line/box is provided, then box your answer
   d. if you use your calculator, give an explanation of what you did on it.
2. The MARK BOX indicates the problems along with their points.
   Check that your copy of the exam has all of the problems.
3. During this test, do not leave your seat.
   If you have a question, raise your hand.
   When you finish: turn your exam over, put your pencil down, raise your hand.
4. This closed book/notes quiz covers (from *Calculus*, by Varberg and Purcell) : Chapter 7.

Problem Source:

1a. § 7.9 sample test # 1
1b. § 7.4 homework problem # 20
1c. § 7.7 lookat problem # 5
1d. § 7.7 lookat problem # 17
2a. § 7.9 sample test # 33
2b. § 7.4 lookat problem # 25
2c. § 7.7 lookat problem # 30
2d. § 7.7 homework problem # 36
3. § 7.4 lookat problem # 32
4. § 7.9 sample test # 40
5. my math 142 exam from fall 93 # 5
6. a variation on § 7.1 lookat problem # 43

1. Four derivatives: **1a** – **1d** .

1a. $\frac{d}{dx} \ln \frac{x^4}{2} =$ $\frac{4}{x}$.

$$= \frac{d}{dx}\left[\ln x^4 \quad + {}^-\ln 2 \right]$$

$$= \frac{d}{dx}\left[4 \ln x \quad + {}^-\ln 2 \right]$$

$$= \frac{4}{x} + 0$$

1b. $\frac{d}{dx} \log_{10}\left(x^3+9\right) =$ $\frac{3x^2}{(x^3+9)\ln 10}$.

$$= \frac{1}{x^3+9} \cdot 3x^2 \cdot \frac{1}{\ln 10}$$

1c. $\frac{d}{dx}\, e^{\tan x} =$ $(\sec^2 x)\ e^{\tan x}$.

$$= e^{\tan x} \cdot D_x \tan x$$

$$= e^{\tan x} \cdot \sec^2 x$$

1d. $\frac{d}{dx} \cos\left(\tan^{-1} x\right) =$ $\dfrac{-x}{(1+x^2)^{3/2}}$.

In **1d**, your answer should NOT involve trig functions.

$$\frac{d}{dx} \cos(\tan^{-1} x) = -\sin(\tan^{-1} x) \cdot D_x \tan^{-1} x$$

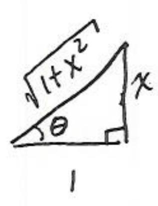
$$= -\sin(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

and :

$\sin(\tan^{-1} x) = \sin\theta = \dfrac{x}{\sqrt{1+x^2}}$ ← If $0 \le \theta < \pi/2$

$\theta = \tan^{-1} x \Rightarrow -\pi/2 < \theta < \pi/2$. then $x \geq 0$

$\tan\theta = x = \dfrac{x}{1}$ and $\sin\theta \geq 0$, good!



← If $-\pi/2 < \theta < 0$

then $x < 0$

and $\sin\theta < 0$ good!

2. Four integral: **2a – 2d** . ⊛ HINT: $+ C$

2a. $\displaystyle\int \frac{-1}{x + x(\ln x)^2}\,dx = \underline{\; -\tan^{-1}(\ln x) + C \;}$.

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$$= -\int \frac{1}{x}\,\frac{1}{1+(\ln x)^2}\,dx$$

$$= -\int \frac{1}{1+u^2}\,du$$

$$= -\tan^{-1} u + C$$

2b. $\displaystyle\int \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx = \underline{\; \frac{2\cdot 5^{\sqrt{x}}}{\ln 5} + C \;}$.

$$\boxed{\begin{aligned} u &= x^{\frac{1}{2}} \\ du &= \tfrac{1}{2}x^{-1/2}\,dx \end{aligned}}$$

$$= \int 5^{x^{1/2}} \cdot x^{-1/2}\,dx$$

$$= 2\int 5^u\,du$$

$$= 2\cdot\frac{5^u}{\ln 5} + C$$

2c. $\int \sin^2 x \, \cos x \, dx =$ $\frac{1}{3} \sin^3 x + C$.

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}}$$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + C$$

2d. $\int \frac{e^x}{1+e^{2x}} \, dx =$ $\tan^{-1} e^x + C$.

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x \, dx \end{aligned}}$$

$$= \int \frac{e^x \, dx}{1+(e^x)^2}$$

$$= \int \frac{du}{1+u^2}$$

$$= \tan^{-1} u + C$$

3. $\frac{d}{dx}\left(\ln x^2\right)^{2x+3} =$ ______________________________.

$$y = \left(\ln x^2\right)^{2x+3}$$

$$\ln y = \ln \left(\ln x^2\right)^{2x+3}$$

$$= (2x+3)\ \ln\left(\ln x^2\right)$$

$$= (2x+3)\ \ln\left(2 \ln x\right).$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \left[D_x\ (2x+3)\right] \cdot \ln(2\ln x) + (2x+3)\left[D_x \ln(2\ln x)\right]$$

$$= 2\ln(2\ln x) + (2x+3)\cdot\frac{1}{2\ln x}\cdot D_x\,(2\ln x)$$

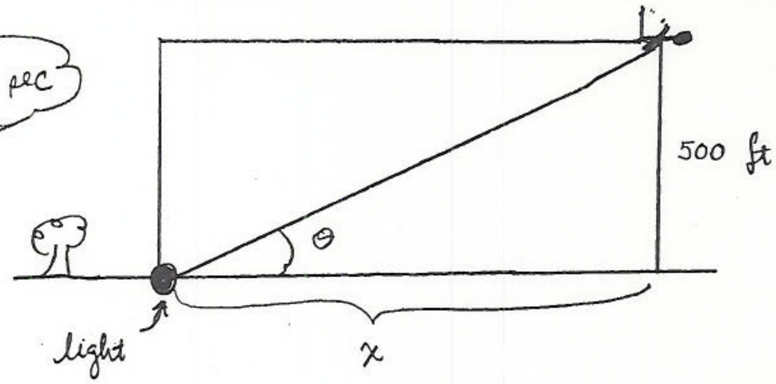
$$= 2\ln(2\ln x) + \frac{2x+3}{\cancel{2}\ln x}\cdot\frac{\cancel{2}}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\ln x^2\right)^{2x+3}\left[2\ln(\underbrace{2\ln x}_{=\ln x^2}) + \frac{2x+3}{x\ln x}\right]$$

4. An airplane is flying horizontally at an altitude of 500 feet at the speed of 300 feet per second directly away from a seachlight on the ground. The searchlight is kept directed at the plane. At what rate is the angle between the light beam and the ground changing when this angle is 30^o?   Ps: do not forget your units.

ANSWER: The angle is changing at the rate of $-\,^{3}/_{20}\ ^{rad}/_{sec}$.

ft & sec



WTS : $\frac{d\theta}{dt}$ when $\theta = 30^\circ$

given $\frac{dx}{dt} = 300\ \frac{ft}{sec}$

so use $\boxed{\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt}}$

$$\tan\theta = \frac{500}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{500}{x}\right) = \tan^{-1}\left(500x^{-1}\right)$$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{500}{x}\right)^2}\cdot\frac{-500}{x^2} = \frac{x^2}{x^2+(500)^2}\cdot\frac{-500}{x^2} = \frac{-500}{x^2+(500)^2}$$

if $\theta = 30^\circ$, then $x = 500\sqrt{3}$.

$$\boxed{\tan\theta = \frac{500}{x} \Rightarrow x = \frac{500}{\tan\theta} = \frac{500}{\tan 30^\circ} = 500\sqrt{3}}$$

so: when $\theta = 30^\circ$

$$\frac{d\theta}{dt} = \frac{-500}{(500\sqrt{3})^2+(500)^2}\cdot 300 = -\frac{500}{500\left(500\cdot 3+500\right)}\cdot 300$$

$$= -\frac{300}{1500+500} = -\frac{300}{2000} = -\frac{3}{20}$$

Note $-\frac{3}{20}\ \frac{rad}{sec} = -\frac{3}{20}\ \frac{\cancel{rad}}{sec}\cdot\frac{180}{\pi}\ \frac{degree}{\cancel{rad}} = \frac{-3\cdot 180}{20\pi}\ \frac{deg}{sec}$

$$= \frac{-3\cdot 9}{\pi}\ \frac{deg}{sec} = \frac{-27}{\pi}\ ^{deg}/_{sec} .$$

5. In 1921, President Warren G. Harding presentd Marie Curie a gift of 2 gram of radium on behalf of the women of the United States. Using the fact that the half-life of radium is 1656 years, determine how much of the original 2-gram gift is left today (in 1998). Your answer can involve exponentials and logs.

ANSWER:

Today there is ____________, which is approximately ____________, grams left.

Let $t=0$ in 1921. In 1998, $t = 1998 - 1921 = 77$

$$P(t) = P_0 e^{kt} \Rightarrow P(t) = 2 \cdot e^{kt}$$

$$\tfrac{1}{2} \text{ life} = 1656 \text{ yrs} \Rightarrow 1 = P(1656) = 2e^{k \cdot 1656}$$

$$\Rightarrow e^{1656k} = \frac{1}{2}$$

$$\Rightarrow 1656k = \ln\left(\tfrac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$$

$$\Rightarrow \boxed{k = \frac{-\ln 2}{1656}}$$

$$P(t) = 2 \cdot e^{\frac{-\ln 2 \cdot t}{1656}}$$

$$P(77) = 2e^{\frac{-77 \ln 2}{1656}} \approx 1.9366$$

6. Use Riemann sums to calculate the below limit.

$$\lim_{n\to\infty}\left[\frac{1}{n^2+1}+\frac{2}{n^2+4}+\frac{3}{n^2+9}+\frac{4}{n^2+16}+\ldots+\frac{2}{5n}\right] = \underline{\frac{\ln 5}{2}} \approx .8047.$$

I'll get you started:

$$\left[\frac{1}{n^2+1}+\frac{2}{n^2+4}+\frac{3}{n^2+9}+\frac{4}{n^2+16}+\ldots+\frac{2}{5n}\right]$$

$$=\left[\frac{1}{n^2+1}+\frac{2}{n^2+2^2}+\frac{3}{n^2+3^2}+\frac{4}{n^2+4^2}+\ldots+\frac{2n}{5n^2}\right]$$

$$=\left[\frac{1}{n^2+1}+\frac{2}{n^2+2^2}+\frac{3}{n^2+3^2}+\frac{4}{n^2+4^2}+\ldots+\frac{2n}{n^2+4n^2}\right]$$

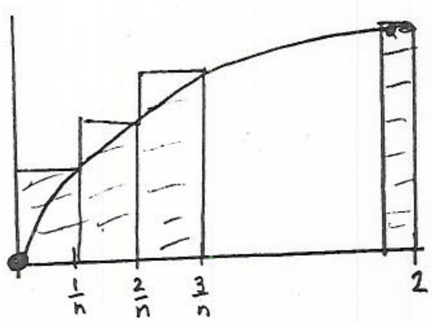
$$=\left[\frac{1}{n+\frac{1}{n}}+\frac{2}{n+\frac{2^2}{n}}+\frac{3}{n+\frac{3^2}{n}}+\frac{4}{n+\frac{4^2}{n}}+\cdots+\frac{2n}{n+4n}\right]\cdot\frac{1}{n}$$ factor n out of denom.

$$=\left[\frac{\frac{1}{n}}{1+\frac{1}{n^2}}+\frac{\frac{2}{n}}{1+\frac{2^2}{n^2}}+\frac{\frac{3}{n}}{1+\frac{3^2}{n^2}}+\frac{\frac{4}{n}}{1+\frac{4^2}{n^2}}+\cdots+\frac{2}{1+4}\right]\cdot\frac{1}{n}$$ divide num. & de by n.

$$=\left[\frac{\frac{1}{n}}{1+(\frac{1}{n})^2}+\frac{\frac{2}{n}}{1+(\frac{2}{n})^2}+\frac{\frac{3}{n}}{1+(\frac{3}{n})^2}+\frac{\frac{4}{n}}{1+(\frac{4}{n})^2}+\cdots+\frac{2}{1+4}\right]\frac{1}{n}$$

Let $f(x)=\frac{x}{1+x^2}$

$$=\left[f(\tfrac{1}{n})+f(\tfrac{2}{n})+f(\tfrac{3}{n})+f(\tfrac{4}{n})+\cdots+f(2)\right]\frac{1}{n}$$



So the limit

$$=\int_0^2 f(x)\,dx=\int_0^2\frac{x}{1+x^2}\,dx$$ $u=1+x^2$, $du=2x\,d$

$$=\frac{1}{2}\int_1^5\frac{du}{u}=\frac{1}{2}\ln u\Big|_1^5=\frac{1}{2}\left[\ln 5-\ln 1\right]$$

more space provided ⟹