If you do not make at least a 50% on this exam's *first few problems*, then your **total** exam score will be only as many points was you managed to get on the *first few problems*. Here are some typical *first few problems*.

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1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: You should NOT write the words absolute nor conditional on Problem 1!

1a. Sequences (Afterall, this is needed for Geometric Series!)

Let $-\infty < r < \infty$. (Fill-in-the blanks with *exists* or *does not exist, i.e. DNE*)

- If $|r| < 1$, then $\lim_{n\to\infty} r^n$ exists (=0)
- If $|r| > 1$, then $\lim_{n\to\infty} r^n$ DNE
- If $r = 1$, then $\lim_{n\to\infty} r^n$ exists (= 1)
- If $r =$ -1, then $\lim_{n\to\infty} r^n$ DNE (osc.)

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ <1
- diverges if and only if $|r|$ ≥ 1

1c. ***p*-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p >1
- diverges if and only if p ≤ 1

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a decreasing (or non increasing) function .

Then $\sum a_n$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges

1g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If ρ <1 then $\sum a_n$ converges.
- If ρ >1 then $\sum a_n$ diverges.
- If ρ $=1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.
If

- a_n $\underline{\quad > \quad}$ a_{n+1} for each $n \in \mathbb{N}$ (decreasing)
- $\lim_{n\to\infty} a_n = \underline{\quad 0 \quad}$

then $\sum(-1)^n a_n$ converges

1i. n^{th}**-term test** for an arbitrary series $\sum a_n$.
If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges .

1j. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1k. If a power series in $x - x_0$ has radius of convergence R where $0 < R < \infty$, then the power series is:

- absolutely convergent for $x_0 - R < x < x_0 + R$
- divergent for $x < x_0 - R$ and $x_0 + R < x$

1l. Consider a **function** $y = f(x)$ where $f\colon [1,\infty) \to \mathbb{R}$.
Next consider the corresponding **sequence** $\{a_n\}_{n=1}^{\infty}$ where $a_n \stackrel{\text{def.}}{=} f(n)$.

- If the limit of the **function** $y = f(x)$ as $x \to \infty$ is L,

  then the limit of the corresponding **sequence** $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$ is L .
- If $\lim_{n\to\infty} a_n = L$, is it necessarily true that $\lim_{x\to\infty} f(x) = L$? Circle: **Yes** or **No**

2. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=17}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge?
Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$?

if NO $\Rightarrow$ $\sum a_n$ is divergent

if YES $\Downarrow$ (from "Does $\sum |a_n|$ converge?")

$\sum a_n$ is absolutely convergent

if YES $\Downarrow$ (from "Does $\lim_{n\to\infty} |a_n| = 0$?")

Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is Conditionally Convergent

3. Circle T if the statement is TRUE. Circle F if the statement if FALSE. To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true. Scoring: 2 pts for a correct answer, 1 pt for a blank answer, 0 pts for an incorrect answer.

T F — If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

T F — If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges. $\left(\text{eg}, a_n = \frac{1}{n}\right)$

T F — If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty} a_n = L$ and $f\colon [0,\infty) \to \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number then $\lim_{x\to\infty} f(x) = L$. Consider $f(x) = \sin(\pi x)$.

T F — If a function $f\colon [0,\infty) \to \mathbb{R}$ satisfies that $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{n\to\infty} a_n = L$.

T F — If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

T F — If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge. e.g. $\begin{cases} a_n = \frac{1}{n} \\ b_n = -\frac{1}{n} \end{cases}$

T F — If $r \neq 1$ and $S_N = \sum_{n=17}^{N} r^n$, then $S_N = \dfrac{r^{17} - r^{N+1}}{1-r}$ for each $N > 17$.

NOTICE, the above sum starts at $n = 17$, not at $n = 0$.

$$1 S_N = r^{17} + r^{18} + r^{19} + \cdots + r^N$$

$$r S_N = r^{18} + r^{19} + \cdots + r^N + r^{N+1}$$

substract

$$(1-r) S_N = r^{17} - r^{N+1}$$

MORE ⇒

4. Geometric Series. Let, for $N \geq 102$,

$$s_N = \sum_{n=102}^{N} 3\frac{2^n}{7^{n+1}} .$$

4a. Do some algebra to write s_N as $\sum_{n=102}^{N} c\, r^n$ for an appropriate constant c and ratio r.

$$\boxed{s_N = \sum_{n=102}^{N} \frac{3}{7}\left(\frac{2}{7}\right)^n}$$

$$3\frac{2^n}{7^{n+1}} = 3\cdot\frac{2^n}{7^n\cdot 7^1} = \frac{3}{7}\,\frac{2^n}{7^n} = \frac{3}{7}\left(\frac{2}{7}\right)^n$$

4b. Using the method from class (rather than some formula), find an expression for s_N in closed form (i.e. without a summation $\sum$ sign nor some dots $\ldots$.).

$$\boxed{s_N = \frac{3}{5}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]}$$

$$1\, s_N = \frac{3}{7}\left[\left(\frac{2}{7}\right)^{102} + \left(\frac{2}{7}\right)^{103} + \cdots + \left(\frac{2}{7}\right)^{N}\right] \overset{\text{or}}{=} \frac{3}{7}\left(\frac{2}{7}\right)^{102} + \frac{3}{7}\left(\frac{2}{7}\right)^{103} + \cdots + \frac{3}{7}\left(\frac{2}{7}\right)^{N}$$

$$\frac{2}{7}\, s_N = \frac{3}{7}\left[\left(\frac{2}{7}\right)^{103} + \cdots + \left(\frac{2}{7}\right)^{N} + \left(\frac{2}{7}\right)^{N+1}\right] \overset{\text{or}}{=} \frac{3}{7}\left(\frac{2}{7}\right)^{103} + \cdots + \frac{3}{7}\left(\frac{2}{7}\right)^{N} + \frac{3}{7}\left(\frac{2}{7}\right)^{N+1}$$

$$\left(1-\frac{2}{7}\right)s_N = \frac{3}{7}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right] \overset{\text{or}}{=} \frac{3}{7}\left(\frac{2}{7}\right)^{102} - \frac{3}{7}\left(\frac{2}{7}\right)^{N+1}$$

$$\underset{\frac{5}{7}}{\parallel} \implies s_N = \frac{7}{5}\cdot\frac{3}{7}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]$$

4c. Does $\sum_{n=102}^{\infty} 3\frac{2^n}{7^{n+1}}$ converge or diverge? If it converges, find its sum. Justify your answer.

$$\boxed{\sum_{n=102}^{\infty} 3\frac{2^n}{7^{n+1}} = \frac{3}{5}\left(\frac{2}{7}\right)^{102}}$$

Geometric Series, ratio $r = \frac{2}{7}$, $|r|<1$ $\Rightarrow$ converges

$$\sum_{n=102}^{\infty} 3\cdot\frac{2^n}{7^{n+1}} \overset{\text{def}}{=} \lim_{N\to\infty} \sum_{n=102}^{N} 3\cdot\frac{2^n}{7^{n+1}} \overset{(4b)}{=} \lim_{N\to\infty} \frac{3}{5}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]$$

$$\overset{(1a)}{=} \frac{3}{5}\left[\left(\frac{2}{7}\right)^{102} - 0\right]$$