Prof. Girardi | Math 142 | Fall 2009 | 11.12.09 | Exam 2

MARK BOX		
PROBLEM	POINTS	
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
take home	10	
%	100	

NAME: James Bond

CLASS PIN: 007

INSTRUCTIONS:

(1) To receive credit you must:
   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
   (b) if a line/box is provided, then:
   — show you work BELOW the line/box
   — put your answer on/in the line/box
   (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8th ed.):

   (a) Sections 10.1 - 10.6, 10.8 for the inclass problems
   (b) whole of Ch 10 for in class fill-in-blank and true/false problems
   (c) Section 10.7. 10.9, 10.10 for the take home part.

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Problem Inspiration: See the answer key.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the above Instructions.

Signature : ____________________

1&2. Spring 06, Exam 3 and Fall 08, Exam 2
3. Spring 06, Exam 3
4. Serious Series Problem #2
5. Maple Lab homework # 9.
6. Spring 06, Exam 3
7. HMWK 10.8 #41
8. HMWK 10.8 # 35

1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1a. Sequences Let $-\infty < r < \infty$. (Fill-in-the blanks with *exists* or *does not exist, i.e. DNE*)

- If $|r| < 1$, then $\lim_{n\to\infty} r^n$ exists
- If $|r| > 1$, then $\lim_{n\to\infty} r^n$ DNE
- If $r = 1$, then $\lim_{n\to\infty} r^n$ exists
- If $r = -1$, then $\lim_{n\to\infty} r^n$ DNE

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

1c. p-series where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a decreasing (or nonincreasing) function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

1g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$ (decreasing)
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

1i. n^{th}-term test for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges .

1j. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1k. If a power series in $x - x_0$ has radius of convergence R where $0 < R < \infty$, then the power series is:

- absolutely convergent for $x \in (x_0 - R,\ x_0 + R)$
- divergent for $x < x_0 - R$ and $x_0 - R < x$.

for 1l - 1o

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = p_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = p_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

1l. In open form (i.e., with … and without a $\sum$-sign)

$$p_N(x) = f^{0}(x_0) + f^{(1)}(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{N}(x_0)}{N!}(x-x_0)^N$$

1m. In closed form (i.e., with a $\sum$-sign and without …)

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1n. In closed form (i.e., with a $\sum$-sign and without …)

$$p_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1o. We know that $f(x) = p_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$R_N(x) = \boxed{\dfrac{f^{N+1}(c)}{(N+1)!}(x-x_0)^{N+1}}$ for some c between $\boxed{x}$ and $\boxed{x_0}$.

1p. Did you write your PIN on the cover page (under your name)? It's worth 5 points. ☺

2. Circle T if the statement is TRUE. Circle F if the statement if FALSE. To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true. Scoring: 2 pts for a correct answer, 1 pt for a blank answer, 0 pts for an incorrect answer.

HMWK Ch 10 Review 9 c/d

T F If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty} a_n = L$ and $f\colon [0,\infty) \to \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number then $\lim_{x\to\infty} f(x) = L$. Consider $f(x) = \sin(2\pi x)$

T F If a function $f\colon [0,\infty) \to \mathbb{R}$ satisfies that $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{n\to\infty} a_n = L$.

Thm 10.4.3 T F If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

$a_n = \frac{1}{n}$, $b_n = \frac{-1}{n}$ T F If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

T F If $r \neq 1$ and $S_N = \sum_{n=17}^{N} r^n$, then $S_N = \dfrac{r^{17} - r^{N+1}}{1-r}$ for each $N > 17$.

NOTICE, the above sum starts at $n = 17$, not at $n = 0$.

$$S_N = r^{17} + r^{18} + \cdots + r^N$$
$$r\,S_N = \quad r^{18} + \cdots + r^N + r^{N+1}$$
$$(1-r)\,S_N = r^{17} - r^{N+1}$$

3. For the following **SEQUENCES**:

- if the limit exists, find it
- if the limit does not exist, then say that it DNE.

Put your ANSWER IN the box and show your WORK BELOW the box.

3a.

$$\lim_{n\to\infty} \frac{(4n+1)(5n+2)}{17n^2} = \frac{20}{17}$$

$$\frac{(4n+1)(5n+2)}{17n^2} = \frac{20n^2 + (\text{who cares})\,n + (\text{some constant})}{17n^2}$$

3b.

$$\lim_{n\to\infty} (\text{-}1)^n \frac{(4n+1)(5n+2)}{17n^2} = \text{DNE}$$

oscillating

3c.

$$\lim_{n\to\infty} (\ 0.9999999917)^n = 0$$

$$\lim_{n\to\infty} r^n = 0 \quad \text{when} \quad |r| < 1$$

4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

- [ ] absolutely convergent
- [x] conditionally convergent
- [ ] divergent

- abs. conv? consider $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$

  p-series $p = 1/2 < 1 \Rightarrow$ diverges

  So <u>not</u> abs. conv.

- cond. conv? Consider $\sum \frac{(-1)^n}{\sqrt{n}} = \sum (-1)^n \frac{1}{\sqrt{n}}$

A.S.T. with $u_n = \frac{1}{\sqrt{n}}$.

① u_n's are decreasing since $u_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = u_{n+1}$

② $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

So by A.S.T., $\sum (-1)^n \frac{1}{\sqrt{n}}$ converges

5. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$$

- [x] absolutely convergent
- [ ] ~~conditionally convergent~~ b/c $\frac{2^n 3^n}{n^n} > 0$.
- [ ] divergent

Hint: $\frac{2^n 3^n}{n^n} = \left(\frac{2*3}{n}\right)^n$

Root Test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{2\cdot 3}{n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{6}{n} = 0 < 1$$

$\Downarrow$

converges

6. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=17}^{\infty} \frac{1}{(\ln n)^4}$$

- [ ] absolutely convergent
- [ ] ~~conditionally convergent~~ b/c
- [x] divergent

Hint: For any $0 < q < \infty$, if n is big enough then $\ln n < n^q$ and so $\frac{1}{(n^q)^4} < \frac{1}{(\ln n)^4}$.

$$\frac{1}{(\ln n)^4} \overset{q=\frac{1}{4}}{>} \frac{1}{(n^{1/4})^4} = \frac{1}{n}$$

$\sum \frac{1}{n}$ diverges, p-series, $p=1 \leq 1$.

CT $\Rightarrow \sum \frac{1}{(\ln n)^4}$ diverges

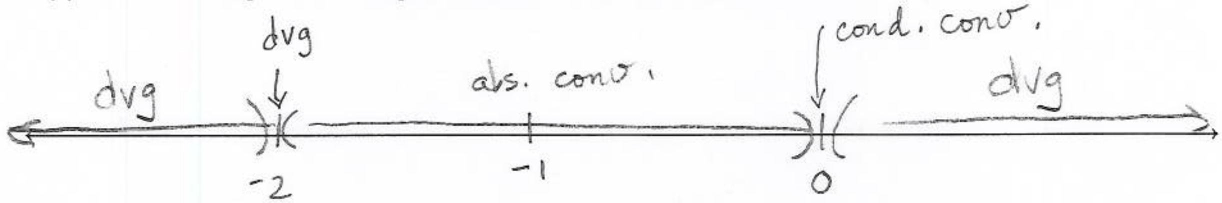
$(x+1)^n = (x - {}^-1)^n$

7. Consider the formal power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x+1)^n}{n}$$

The center is $x_0 =$ -1 and the radius of convergence is $R =$ 1.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$$\rho = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{n+1} \cdot \frac{n}{(x+1)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} |x+1| = |x+1| \lim_{n\to\infty} \frac{n}{n+1} = |x+1|$$

or

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^n (x+1)^n}{n} \right|^{1/n} = \lim_{n\to\infty} \frac{|x+1|}{n^{1/n}} = |x+1| \lim_{n\to\infty} \frac{1}{n^{1/n}} = |x+1|$$

abs conv when $|x+1| < 1$

Check endpts

$x = -2$ $\quad \sum (-1)^n \frac{(-2+1)^n}{n} = \sum \frac{(-1)^n(-1)^n}{n} = \sum \frac{1}{n}$ dvg p-series $p=1$.

$x = 0$ $\quad \sum \frac{(-1)^n (0+1)^n}{n} = \sum \frac{(-1)^n}{n}$. ← converges by AST

$\frac{1}{n} \to 0$

$\frac{1}{n} > \frac{1}{n+1}$

So $\sum \frac{(-1)^n}{n}$ is cond. conv.

8a ~~7a~~. Let

$$a_n = \frac{x^{2n+1}}{(2n+1)!}$$

Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{x^2}{(2n+2)(2n+3)}}$$

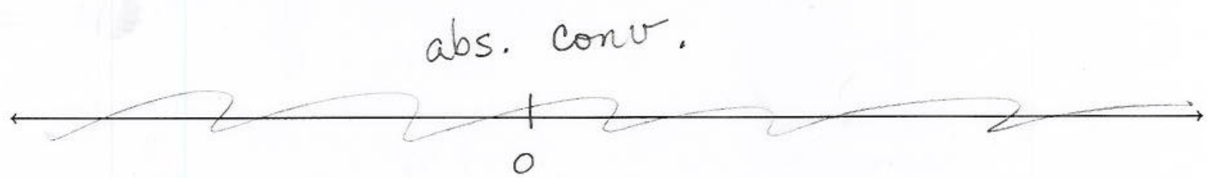
$$\frac{a_{n+1}}{a_n} = \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} = \frac{x^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{x^{2n+1}}$$

$$= \frac{x^{2n+3}}{x^{2n+1}} \frac{(2n+1)!}{(2n+3)!} = \frac{x^{2n+1} \cdot x^2}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+1)!\,(2n+2)(2n+3)}$$

8b ~~7b~~. Consider the formal power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

The center is $x_0 =$ ____0____ and the radius of convergence is $R =$ ____∞____.
As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)}$$

$$= |x|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = |x|^2 \cdot 0 < 1$$

always :)