

MARK BOX		
PROBLEM	POINTS	
a - j	10	
TOTAL	10	

NAME (legibly printed): James Bond

class PIN: 007

INSTRUCTIONS:

- (1) To receive credit you must:
 - (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that just appears;**
such explanations help with partial credit
 - (b) if a line/box is provided, then:
 - show you work **BELOW** the line/box
 - put your answer on/in the line/box
 - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
- (3) This exam covers (from *Calculus* by Anton, Bivens, Davis 8th ed.): § 10.7, 10.9, 10.10 .

Problem Inspiration: just like the homework.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Furthermore, I have not only read but will also follow the above Instructions.

I hereby verify that I did NOT receive help from other people on this take-home exam problem.

Signature : _____

**Due Friday November 13 by 1pm.
Either hand me your paper in class or
slip your paper under my office (LC 309C) door.**



Just FYI

Homework on Taylor/Maclaurin Polynomials and Series

Part 1 — Fill in the box

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0 .

Let $y = P_N(x)$ be the N^{th} -order Taylor polynomial of $y = f(x)$ about x_0 .

Let $y = R_N(x)$ be the N^{th} -order Taylor remainder of $y = f(x)$ about x_0 .

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0 .

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0 .

A. In open form (i.e., with ... and without a \sum -sign)

$$P_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \dots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N$$

B. In closed form (i.e., with a \sum -sign and without ...)

$$P_N(x) = \sum_{n=0}^N \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

C. In open form (i.e., with ... and without a \sum -sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \dots$$

D. In closed form (i.e., with a \sum -sign and without ...)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

E. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x - x_0)^{(N+1)}$$
 for some c between x and x_0 .

F. The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

1.5

FYI

Taylor/Maclaurin Polynomials and Series

Do parts (a) - (j) for the following problem.

$$f(x) = (7-x)^{-2} \quad x_0 = 5 \quad J = (4, 6) .$$

You might find it easier to do problems (a) - (j) in a different order. Just do what you find easiest.

- On parts (a) - (i), use ideas from only Sections 10.7 and 10.9, i.e., use only:
 - the definition of Taylor polynomial
 - the definition of Taylor series
 - the theorem/error-estimate on the N^{th} -Remainder term for Taylor polynomials.
 Do **NOT** use a known Taylor Series (i.e., do not use methods from Section 10.10).
- On part (j), the very last part, use a known Taylor Series (as from the handout **Commonly Used Taylor Series**) and methods from Section 10.10.

a. Find the following. Note the first column are functions of x and the second column are numbers.

$f^{(0)}(x) = (7-x)^{-2} = 1! (7-x)^{-2}$	$f^{(0)}(x_0) = (2)^{-2} \equiv \frac{1}{4}$
$f^{(1)}(x) = 2 (7-x)^{-3} = 2! (7-x)^{-3}$	$f^{(1)}(x_0) = 2! (2)^{-3} \equiv \frac{2}{2^3} \equiv \frac{1}{4}$
$f^{(2)}(x) = 2 \cdot 3 (7-x)^{-4} = 3! (7-x)^{-4}$	$f^{(2)}(x_0) = 3! (2)^{-4}$
$f^{(3)}(x) = 2 \cdot 3 \cdot 4 (7-x)^{-5} = 4! (7-x)^{-5}$	$f^{(3)}(x_0) = 4! (2)^{-5}$
$f^{(4)}(x) = 2 \cdot 3 \cdot 4 \cdot 5 (7-x)^{-6} = 5! (7-x)^{-6}$	$f^{(4)}(x_0) = 5! (2)^{-6}$

b. Find N^{th} -order Taylor polynomial of $y = f(x)$ about x_0 in OPEN form for $N = 0, 1, 2, 3, 4$. C_n

$P_0(x) = \frac{1}{4}$	$C_0 = \frac{1}{4} = \frac{1}{2^2}$
$P_1(x) = \frac{1}{4} + \frac{1}{4} (x-5)$	$C_1 = \frac{1}{4} = \frac{2}{2^3}$
$P_2(x) = \frac{1}{4} + \frac{1}{4} (x-5) + \frac{3}{2^4} (x-5)^2$	$C_2 = \frac{3!}{2!} \frac{1}{2^4} = \frac{3}{2^4}$
$P_3(x) = \frac{1}{4} + \frac{1}{4} (x-5) + \frac{3}{2^4} (x-5)^2 + \frac{4}{2^5} (x-5)^3$	$C_3 = \frac{4!}{3!} \frac{1}{2^5} = \frac{4}{2^5}$
$P_4(x) = \frac{1}{4} + \frac{1}{4} (x-5) + \frac{3}{2^4} (x-5)^2 + \frac{4}{2^5} (x-5)^3 + \frac{5}{2^6} (x-5)^4$	$C_4 = \frac{5!}{4!} \frac{1}{2^6} = \frac{5}{2^6}$

c. Find the Taylor series of $y = f(x)$ about x_0 in OPEN form.

$$P_{\infty}(x) = \frac{1}{4} + \frac{1}{4}(x-5) + \frac{3}{2^4}(x-5)^2 + \frac{4}{2^5}(x-5)^3 + \frac{5}{2^6}(x-5)^4 + \dots$$

d. Find the Taylor series of $y = f(x)$ about x_0 in CLOSED form.

$$P_{\infty}(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} (x-5)^n$$

e. Find the n^{th} Taylor coefficient of $y = f(x)$ about x_0 .

$$c_n = \frac{n+1}{2^{n+2}} \quad \text{or} \quad (n+1) 2^{-(n+2)}$$

f. Find the interval I of convergence of the Taylor series $y = f(x)$ about x_0 . Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

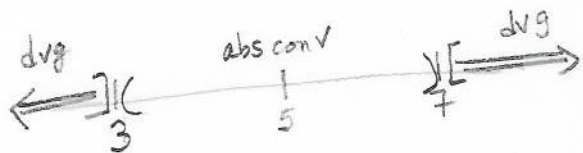
$$I = (3, 7)$$

Ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)(x-5)^{n+1}}{2^{n+3}} \cdot \frac{2^{n+2}}{(n+1)(x-5)^n} \right| =$$

$$\lim_{n \rightarrow \infty} \frac{n+2}{n+1} \cdot \frac{2^{n+2}}{2^{n+3}} \cdot \left| \frac{(x-5)^{n+1}}{(x-5)^n} \right| = \lim_{n \rightarrow \infty} \frac{n+2}{n+1} \cdot \frac{1}{2} |x-5| =$$

$$\frac{|x-5|}{2} \lim_{n \rightarrow \infty} \frac{n+2}{n+1} = \frac{|x-5|}{2} < 1 \iff |x-5| < 2$$



check endpts

$$x=7 \Rightarrow \sum \frac{n+1}{2^{n+2}} (x-5)^n = \sum \frac{n+1}{2^{n+2}} \cdot 2^n = \frac{1}{4} \sum (n+1) \quad \text{div! (divg!!)}$$

$$x=3 \Rightarrow \sum \frac{n+1}{2^{n+2}} (x-5)^n = \sum \frac{n+1}{2^{n+2}} (-2)^n = \sum (n+1) \frac{1}{2^{n+2}} (-1)^n (2)^n$$

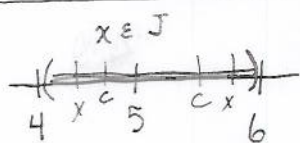
$$= \frac{1}{4} \sum (n+1) (-1)^n \quad \text{divg by } n^{\text{th}} \text{ term test}$$

- g. Consider the given interval J and fix an $N \in \mathbb{N}$. Find a good upper bound for the maximum of $|f^{(N+1)}(c)|$ on the interval J . Your answer can have an N in it but it cannot have an: x, x_0, c . (Note that J is a subset of I but Prof. G. might have picked a smaller J than I to make the problem easier.)

$$\max_{c \in J} |f^{(N+1)}(c)| \leq (N+2)!$$

$$f^N(x) = (N+1)! (7-x)^{-(N+2)}$$

$$|f^{N+1}(c)| = (N+2)! |7-c|^{-(N+3)} = \frac{(N+2)!}{|c-7|^{N+3}}$$



$x \in J$. c is between x & $5 \leftarrow x_0 \Rightarrow c \in J$

$$\Rightarrow 4 < c < 6 \Rightarrow -3 < c-7 < -1 \Rightarrow 1 < |c-7| < 3$$

So $|f^{N+1}(c)| = \frac{(N+2)!}{|c-7|^{N+3}} \leq \frac{(N+2)!}{1^{N+3}} = (N+2)!$

- h. Consider the given interval J and fix an $N \in \mathbb{N}$. For each $x \in J$, find a good upper bound for the maximum of $|R_N(x)|$. Your answer can have an N and x in it but it cannot have an: x_0, c .

$$|R_N(x)| \leq (N+2) |x-5|^{N+1}$$

$$R_N(x) = \frac{f^{N+1}(c)}{(N+1)!} (x-5)^{N+1}$$

for some c btw. x and 5

$$|R_N(x)| = \frac{|f^{N+1}(c)| \cdot |x-5|^{N+1}}{(N+1)!} \stackrel{(g)}{\leq} \frac{(N+2)!}{(N+1)!} |x-5|^{N+1}$$

- i. Carefully show that $f(x) = P_\infty(x)$ for each x in the given interval J by using part (h) and showing that $\lim_{N \rightarrow \infty} |R_N(x)| = 0$ for each $x \in J$.

Fix $x \in J = (4, 6)$.

So $4 < x < 6$ and so $-1 < x-5 < 1$.

Thus $\lim_{N \rightarrow \infty} |x-5|^{N+1} = 0$ since $\lim_{N \rightarrow \infty} r^N = 0$ if $|r| < 1$.

Compute

$$\lim_{N \rightarrow \infty} (N+2) |x-5|^{N+1} \quad \begin{array}{l} \infty \cdot 0 \\ \text{Algebra} \end{array} \quad \lim_{N \rightarrow \infty} \frac{N+2}{|x-5|^{-(N+1)}} \quad \begin{array}{l} \frac{\infty}{\infty} \\ \text{L'H} \end{array}$$

$$\lim_{N \rightarrow \infty} \frac{1}{-(N+1) |x-5|^{-N-1-1}} \quad \begin{array}{l} \text{algebra} \end{array} \quad \lim_{N \rightarrow \infty} \frac{1}{-(N+1)} \cdot \frac{1}{|x-5|^{-(N+2)}} \quad \begin{array}{l} \text{algebra} \end{array}$$

$$\lim_{N \rightarrow \infty} \frac{1}{-(N+1)} \frac{|x-5|^{N+2}}{1} = 0 \cdot 0 = 0.$$

Since $|R_N(x)| \leq (N+2) |x-5|^{N+1}$

and $\lim_{N \rightarrow \infty} (N+2) |x-5|^{N+1} = 0,$

$$\lim_{N \rightarrow \infty} |R_N(x)| = 0.$$

- j. Using a known Taylor Series (as from the handout **Commonly Used Taylor Series**) and methods from Section 10.10, find a power series expansion (in **CLOSED** form) for $y = f(x)$ about x_0 . Also, say when this power series expansion is valid by examining when the **Commonly Used Taylor Series** is valid. **Show all your work and work in a logical fashion.**

$$f(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} (x-5)^n \quad \text{which is valid for } |x-5| < 2$$

i.e. $3 < x < 7$

Hints:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} = (1-r)^{-1} \quad \text{valid when } |r| < 1 \quad (1)$$

$$\frac{1}{7-x} = \frac{1}{2-(x-5)} = \frac{1}{2} \left[\frac{1}{1-\left(\frac{x-5}{2}\right)} \right] \quad (2)$$

$$D_x \frac{1}{7-x} = D_x (7-x)^{-1} = (7-x)^{-2} \quad (3)$$

$$(1) + (2) \Rightarrow \frac{1}{7-x} \stackrel{(*)}{=} \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x-5}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(x-5)^n}{2^{n+1}}$$

$$\text{valid} \Leftrightarrow \left| \frac{x-5}{2} \right| < 1 \Leftrightarrow |x-5| < 2 \Leftrightarrow 3 < x < 7$$

$$(7-x)^{-2} \stackrel{(3)}{=} D_x \frac{1}{7-x} = D_x \left(\sum_{n=0}^{\infty} \frac{(x-5)^n}{2^{n+1}} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} D_x (x-5)^n = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} D_x (x-5)^n$$

why? \uparrow

$$= \sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{2^{n+1}} = \frac{(x-5)^0}{2^2} + \frac{2(x-5)^1}{2^3} + \frac{3(x-5)^2}{2^4} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} (x-5)^n$$

Compare with your solⁿ to part d!