Prof. Girardi	Math 142	Fall 2008	10.30.08	Exam 2

MARK BOX		
PROBLEM	POINTS	
1 a–j	30	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
take home	10	
%	100	

NAME (legibly printed): ______________________

class PIN: 007

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) if a line/box is provided, then:
    — show you work BELOW the line/box
    — put your answer on/in the line/box
  (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.):
Sections 10.1 - 10.6, 10.8 for the inclass and Section 10.7. 10.9, 10.10 for the take home. .

Problem Inspiration: See the answer key.

1 + 2 previous exam: Fall 2006 Exam 3 part 2 # 1 and # 2
homework Ch 10 Review # 1-9.

3. previous exams (3a-3c) and HMWK § 10.1 # 21 (3d)

4. Serious Series' Problem # 8.

5. Example from Class

6. Serious Series' Problem # 14

7. Homework problem § 10.10 # 48.

1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1a. n^{th}**-term test** for an arbitrary series $\sum a_n$.
If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges .

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

1c. p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a nonincreasing (or decreasing) function .

Then $\sum a_n$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.
If $0 < L < \infty$, then $\sum a_n$ converges if and only if Σb_n converges .

1g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.
If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

Ch 10 Review # 7.

1i. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1j. If a power series in $x - x_0$ has radius of convergence R where $0 < R < \infty$, then the power series is:

- absolutely convergent for $x \in (x_0 - R, x_0 + R)$ ... i.e... $x_0 - R < x < x_0 + R$. ie $|x - x_0| < R$
- divergent for $x \notin [x_0 - R, x_0 + R]$ .. i.e. $|x - x_0| > R$

2. Circle T if the statement is TRUE. Circle F if the statement if FALSE. To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true.
Scoreing: 2 pts for a correct answer, 1 pt for a blank answer, 0 pts for an incorrect answer.

Ch 10 Review # 9 c/d

T   F   If a function $f\colon [0,\infty) \to \mathbb{R}$ satisfies that $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{n\to\infty} a_n = L$.

T   F   If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty} a_n = L$ and $f\colon [0,\infty) \to \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{x\to\infty} f(x) = L$. eg... $f(x) = \cos(2\pi x)$

Thm 10.4.3 $\rightarrow$ T   F   If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

$a_n = \frac{1}{n}$, $b_n = -\frac{1}{n}$ $\rightarrow$ T   F   If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

T   F   If $S_N = \sum_{n=17}^{N} r^n$, then $S_N = \frac{r^{17} - r^{N+1}}{1-r}$ for each $N > 17$.

$r \neq 1$ and

NOTICE, the above sum starts at $n = 17$, not at $n = 0$.

$$S_N = r^{17} + r^{18} + \dots + r^N$$
$$r\,S_N = \qquad r^{18} + \dots + r^N + r^{N+1}$$
$$(1-r)\,S_N = r^{17} - r^{N+1}$$
$$S_N = \frac{r^{17} - r^{N+1}}{1-r}$$

3. Find the limit of the following **sequences**. Indicate your reasoning.
Put ANSWER IN BOX and show WORK BELOW BOX.
There is: 3a, 3b, 3c (on this page), and then 3d on the next page.

3a. $\boxed{\lim_{n\to\infty} (.999917)^n = 0}$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & |r|<1 \\ \text{DNE} & |r|>1 \\ 1 & r=1 \\ \text{DNE} & r=-1. \end{cases}$$

3b. $\boxed{\lim_{n\to\infty} (1.0000017)^n = \infty \text{ or DNE or diverges}}$

3c. $\boxed{\lim_{n\to\infty} \frac{5n^3+6n+3}{17n^3+9n^2+4} = \frac{5}{17}}$

$$\lim_{n\to\infty} \frac{5n^3+6n+3}{17n^3+9n^2+4} = \lim_{n\to\infty} \frac{5+\frac{6}{n^2}+\frac{3}{n^3}}{17+\frac{9}{n}+\frac{4}{n^3}} = \frac{5+0+0}{17+0+0}$$

divide numerator & denom. by $n^{\text{(highest power you see)}} = n^3$.

3d IS ON NEXT THE PAGE ⟹

$$\frac{d}{dn}\left(\frac{n+6}{n+1}\right) = \frac{d}{dn}\left(\frac{n+1 \quad + 5}{n+1}\right)$$

$$= \frac{d}{dn}\left(1 + 5(n+1)^{-1}\right) = -5(n+1)^{-2}$$

3d. $\boxed{\lim_{n\to\infty}\left(\frac{n+6}{n+1}\right)^n \quad = \quad e^{5}}$

Hint: if $\ln(a_n) \to 17$, then $a_n \equiv e^{\ln(a_n)} \to e^{17}$

$$\left(\frac{n+6}{n+1}\right)^n = \left(\frac{1+\frac{6}{n}}{1+\frac{1}{n}}\right)^n \xrightarrow{n\to\infty} \cancel{1^\infty} \quad \text{indeterminate form.}$$

$$\lim_{n\to\infty} \ln\left(\frac{n+6}{n+1}\right)^n = \lim_{n\to\infty} n \ln\left(\frac{n+6}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+6}{n+1}\right)}{n^{-1}}$$

$$\underset{L'H}{\overset{\frac{\ln 1 = 0}{0}}{=}} \lim_{n\to\infty} \frac{D_n \ln\left(\frac{n+6}{n+1}\right)}{D_n \; n^{-1}} \overset{(*)}{=} \lim_{n\to\infty} \frac{\frac{n+1}{n+6}\,\frac{-5}{(n+1)^2}}{-n^{-2}}$$

$$\overset{alg}{=} 5 \lim_{n\to\infty} \frac{n^2}{(n+6)(n+1)} = 5 \lim_{n\to\infty} \frac{n^2}{n^2 + ?n + ??}$$

$$= 5 \lim_{n\to\infty} \frac{1}{1 + \frac{?}{n} + \frac{??}{n^2}} = 5 \cdot \frac{1}{1+0+0} = 5$$

see 3c.

$\boxed{\text{Indeterminate Forms - L'Hôpital's Rule}}$

At $x = u$, $\boxed{}$	has the <u>indeterminate form</u> $\boxed{}$	if $\lim\limits_{x \to u} f(x) = \boxed{}$	and $\lim\limits_{x \to u} g(x) = \boxed{}$
(1) $\frac{f(x)}{g(x)}$	$\frac{0}{0}$	0	0
(2) $\frac{f(x)}{g(x)}$	$\frac{\infty}{\infty}$	∞	∞
(3) $f(x) \cdot g(x)$	$0 \cdot \infty$	0	∞
(4) $f(x) - g(x)$	$\infty - \infty$	∞	∞
(5) $[f(x)]^{g(x)}$	0^0	0	0
(6) $[f(x)]^{g(x)}$	∞^0	∞	0
(*) $\boxed{(7)}$ $[f(x)]^{g(x)}$	1^∞	1	∞

HERE: u stands for any of the symbols a, a^-, a^+, $-\infty$, $+\infty$.

$\boxed{\text{L'Hôpital's Rule}}$

(1) and **(2)**

If:

- $\frac{f(x)}{g(x)}$ has the interdeterminate form $\boxed{\frac{0}{0}}$ or $\boxed{\frac{\infty}{\infty}}$ at u

and

- $\lim\limits_{x \to u} \dfrac{f'(x)}{g'(x)}$ exists (i.e. this limit is a finite number or $-\infty$ or ∞)

then

$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)} .$$

(3)

If $f(x) \cdot g(x)$ has the interdeterminate form $\boxed{0 \cdot \infty}$ at u, then rewrite:

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)}$$, which has the interdeterminate form $\boxed{\frac{0}{0}}$ at u

or

$$f(x) \cdot g(x) = \frac{g(x)}{1/f(x)}$$, which has the interdeterminate form $\boxed{\frac{\infty}{\infty}}$ at u

and then apply L'Hôpital's Rule.

(4)

If $f(x) - g(x)$ has the interdeterminate form $\boxed{\infty - \infty}$ at u,

then use algebraic manipulation to convert $f(x) - g(x)$

into a form of the type $\boxed{\frac{0}{0}}$ or $\boxed{\frac{\infty}{\infty}}$

and then apply L'Hôpital's Rule.

(5)

If $[f(x)]^{g(x)}$ has the interdeterminate form $\boxed{0^0}$ at u, then follow these steps:

Let

$$y = [f(x)]^{g(x)} \ .$$

So

$$\ln y = \ln\Big(\ [\ f(x) \ ]^{g(x)} \ \Big) \ .$$

Next, simplify

$$\ln y = [g(x)] \ \cdot \ \ln[f(x)] \ .$$

Note that $\ln y = [g(x)] \ \cdot \ \ln[f(x)]$ has the interdeterminate form $\boxed{0 \ \cdot \ -\infty}$ at u.

Using an appropriate above method (i.e. **(3)**), evaluate

$$\lim_{x \to u} \ \ln y \ \equiv \ L \ .$$

Conclude

$$\lim_{x \to u} \ \ln[f(x)]^{g(x)} \ = \ L \quad \Longrightarrow \quad \lim_{x \to u} [f(x)]^{g(x)} \ = \ e^L \ .$$

(6) and **(7)**

If $[f(x)]^{g(x)}$ has the interdeterminate form $\boxed{\infty^0}$ or $\boxed{1^\infty}$, then proceed similarly as in **(5)** .

Note that $\ln y = [g(x)] \ \cdot \ \ln[f(x)]$ will have the interdeterminate form

(6) $\boxed{0 \ \cdot \ \infty}$ at u

(7) $\boxed{\infty \ \cdot \ 0}$ at u .

4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{17n^2+n-1}$$

- [ ] absolutely convergent
- [ ] conditionally convergent
- [x] divergent

$$\lim_{n\to\infty} (-1)^n \frac{n^2+1}{17n^2+n-1} = \lim_{n\to\infty} (-1)^n \frac{1+\frac{1}{n^2}}{17+\frac{1}{n}-\frac{1}{n^2}} \neq 0$$

see 3c

b/c it's oscillating.

n^{th} term test for divg. $\Rightarrow$ divg.

5. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$$

- [ ] absolutely convergent
- [x] conditionally convergent
- [ ] divergent

See class notes.

Ex 2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

☐ abs. conv.
☐ cond. conv.
☐ divg.

Well This is really a beefed-up (made harder) version of Ex 1. Let's learn from (i.e. compare to) Ex 1.

Ex. 1. $\sum (-1)^n \frac{1}{n} = \sum (-1)^n \tilde{u}_n$

Ex 2 $\sum (-1)^n \frac{n^3}{n^4+1} = \sum (-1)^n u_n$

$$u_n = \frac{n^3}{n^4+1} \overset{n \text{ big}}{\approx} \frac{n^3}{n^4} = \frac{1}{n} = \tilde{u}_n$$

So $\underbrace{\sum (-1)^n u_n}_{}$ should behave like $\sum (-1)^n \tilde{u}_n$.

$\Downarrow$ Ex 1

cond. conv

he should also be cond. conv. $\Longleftarrow$

Now. to justify our above guess.

① abs. conv.? $\longrightarrow \boxed{}$

Consider $\sum \left| (-1)^n \frac{n^3}{n^4+1} \right| = \sum \frac{n^3}{n^4+1}$

$$0 < \frac{n^3}{n^4+1} \overset{n \text{ big}}{\approx} \frac{n^3}{n^4} = \frac{1}{n} = b_n$$

LCT. $\lim_{n\to\infty} \frac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4}{n^4+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^4}} = \frac{1}{1+0} = 1$

$\downarrow$

$0 < 1 < \infty.$

So $\sum \frac{n^3}{n^4+1}$ & $\sum \frac{1}{n}$ "do the same thing.

$\uparrow$ diverges p-series $p = 1 \le 1$

So $\sum \frac{n^3}{n^4+1}$ diverges

(2) cond. conv.? $\longrightarrow \square$

Consider $\sum (-1)^n \frac{n^3}{n^4+1} = \sum (-1)^n u_n$ with $u_n = \frac{n^3}{n^4+1} > 0$.

So it's an alternating series

AST. (1) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n^3}{n^4+1} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^4}} = \frac{0}{1+0} = 0$ ✓

(2) Are the u_n's dearcasing?

or 17 is good enough

$f(x) = \frac{x^3}{x^4+1}$ ← want dec. so want $f'(x) < 0$ for $x \geq 1$

$f'(x) \underset{\text{work}}{\overset{\text{Calc I}}{=}} \frac{x^2(3-x^4)}{(x^4+1)^2} < 0 \iff 3 - x^4 < 0$

$\iff 3 < x^4$

so dec. for n big enough. $\iff 3^{1/4} < |x|$

$\underset{1.3}{\approx}$

AST. $\Rightarrow \sum \frac{(-1)^n n^3}{n^4+1}$ conv.

Conclusion

$$\left.\begin{array}{ll} \sum \left|\frac{(-1)^n n^3}{n^4+1}\right| & \text{divg} \\ \sum \frac{(-1)^n n^3}{n^4+1} & \text{Convg.} \end{array}\right] \Rightarrow \sum (-1)^n \frac{n^3}{n^4+1} \text{ is Conditionally convergent.}$$

6. Let

$$a_n = \frac{3^n\, n!}{(2n)!}$$

6a. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.
Hint: $(2\ (n+1)\ )! = (2n+2)!$

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{3(n+1)}{(2n+1)(2n+2)} \ \overset{\text{or}}{=}\ \frac{3n+3}{4n^2+6n+2}}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{3^n\, n!} = \frac{3^{n+1}}{3^n}\ \frac{(n+1)!}{n!}\ \frac{(2n)!}{(2n+2)!}$$

$$= \frac{3^n \cdot 3}{3^n}\ \frac{n!\,(n+1)}{n!}\ \frac{(2n)!}{(2n)!\,(2n+1)(2n+2)} = \frac{3(n+1)}{(2n+1)(2n+2)}$$

6b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n\ \frac{3^n\, n!}{(2n)!}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

abs. conv? consider $\sum \left| (-1)^n \frac{3^n\, n!}{(2n)!} \right| = \sum \frac{3^n\, n!}{(2n)!}$

Factorial $\Rightarrow$ ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \overset{(6a)}{=} \lim_{n\to\infty} \frac{3n+3}{4n^2+6n+2} \underset{\text{see 3c}}{=} \lim_{n\to\infty} \frac{\frac{3}{n}+\frac{3}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}}$$

$$= \frac{0+0}{4+0+0} = 0 < 1 \Rightarrow \text{conv.}$$

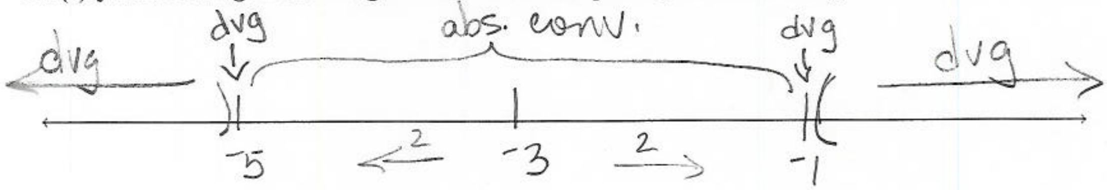
7. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

Hint: $(2x+6)^n = [2(x+3)]^n = 2^n (x+3)^n = 2^n (x-(-3))^n$

The center is $x_0 =$ -3 and the radius of convergence is $R =$ 2.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio Test $\quad \rho = \lim_{n\to\infty} \left| \frac{(2x+6)^{n+1}}{4^{n+1}} \frac{4^n}{(2x+6)^n} \right| = \lim_{n\to\infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\updownarrow$ or

Root Test. $\quad \rho = \lim_{n\to\infty} \left| \frac{(2x+6)^n}{4^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\rho < 1 \iff |2x+6| < 4 \iff 2|x+3| < 4 \iff |x+3| < 2 \iff |x-(-3)| < 2$

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endpts. $\quad -3+2 = -1$ and $-3-2 = -5$

Check endpts

$x = -1$: $\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1 = 1+1+1+1+\dots = \infty$ dvg

$x = -5$ $\quad \sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1\cdot 4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n}$

$= \sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1-1+\dots$

osc btw -1 & $0 \Rightarrow$ dvg