

MARK BOX		
PROBLEM	POINTS	
1 a-j	10	
2	10	
Extra Credit	1	
3a	10	
3b	10	
3c	10	
4	10	
5	10	
6	10	
7	10	
8	10	
%	100	

NAME: Solution Key

SSN: \_\_\_\_\_

please check the box of your section below

 Section 001 (MW 9:05 pm)

or

 Section 002 (MW 10:10 pm)**INSTRUCTIONS:**

- (1) To receive credit you must:
  - (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;**  
such explanations help with partial credit
  - (b) when applicable put your answer on/in the line/box provided
  - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.  
Check that your copy of the exam has all of the problems.
- (3) You may **not** use a calculator, books, personal notes.
- (4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- (5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8<sup>th</sup> ed.):  
Section 7.1 – 7.4, 7.6, 7.7, 8.1 .

**Problem Inspiration:**

1. Math 141 HANDOUT
2. homework problem § 7.1 # 15 and 17 , actually problem § 7.1 # 12
3. homework problems from § 7.2 and 7.3 , homework problem § Ch. 7 Review # 6.
4. homework problem § 7.4 # 28a and 30
5. homework problem § 8.1 # 5
6. homework problem § 8.1 # 25
7. homework problem § 8.1 # 19 and a Prof. Girardi quiz
8. the *ice-cream cone* example from class

1. Fill in the blanks (each worth 1 point).

1a.  $\int \frac{du}{u} = \underline{\ln |u|} + C$

1b. If  $a$  is a constant and  $a > 0$  but  $a \neq 1$ , then

$$\int a^u du = \underline{\frac{a^u}{\ln a}} + C$$

1c.  $\int \sec^2 u du = \underline{\tan u} + C$

1d.  $\int \sec u \tan u du = \underline{\sec u} + C$

1e.  $\int \sin u du = \underline{-\cos u} + C$

1f.  $\int \cot u du = \underline{\ln |\sin u| + C}$  or  $\underline{-\ln |\csc u|} + C$

1g.  $\int \sec u du = \underline{\ln |\sec u + \tan u| + C}$  or  $\underline{-\ln |\sec u - \tan u|} + C$

1h. If  $a$  is a constant and  $a > 0$  then

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \underline{\sin^{-1} \frac{u}{a} + C}$$
 or  $\underline{\arcsin \frac{u}{a}} + C$

1i. If  $a$  is a constant and  $a > 0$  then

$$\int \frac{1}{a^2 + u^2} du = \underline{\frac{1}{a} \tan^{-1} \frac{u}{a} + C}$$
 or  $\underline{\frac{1}{a} \arctan \frac{u}{a}} + C$

1j. The integral of  $y = f(x)$  with respect to  $x$  is denoted by  $\int f(x) dx$ .

The integral of  $x = g(y)$  with respect to  $y$  is denoted by  $\int g(y) dy$ .

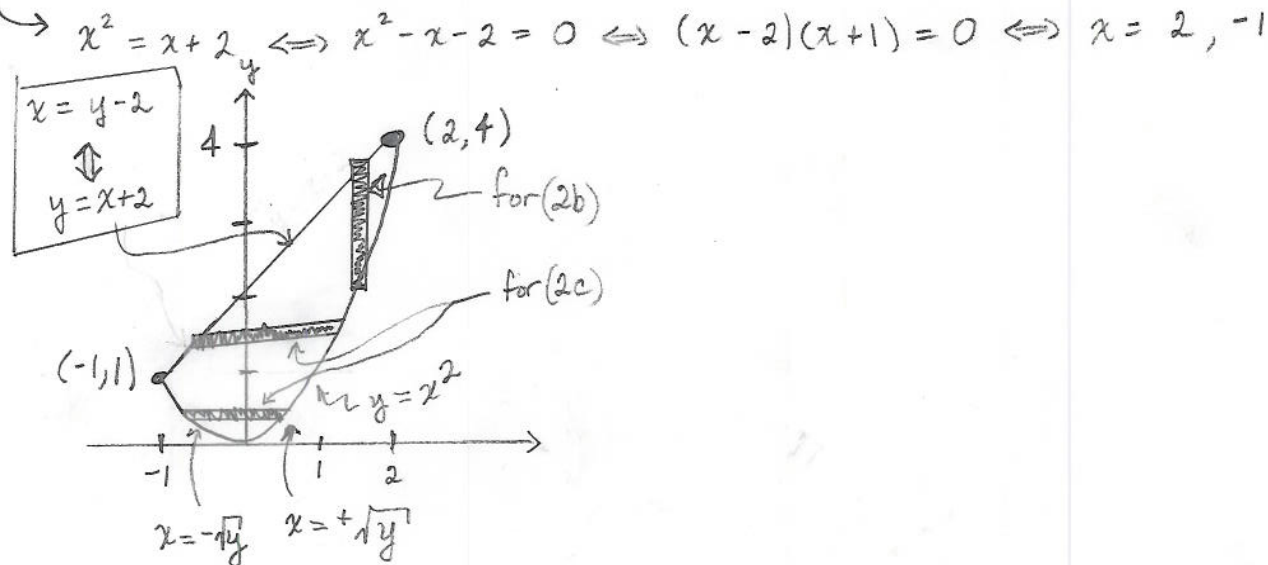
2. Let  $R$  be the region enclosed by

$$y = x^2 \quad \text{and} \quad y = x + 2.$$

Let  $A$  be the area of the region  $R$ .

2a. The points of intersection of  $y = x^2$  and  $y = x + 2$  are  $P = (-1, 1)$  and  $Q = (2, 4)$ .

Make a rough sketch of the region  $R$ , labeling  $P$  and  $Q$ .



2b. Express the area  $A$  as integral(s) with respect to  $x$  (so you want  $dx$ ).

You do NOT have to evaluate the integral(s) nor do lots of algebra.

$$A = \int_{x=-1}^{x=2} [(x+2) - (x^2)] dx \quad \text{or} \quad \int_{x=-1}^{x=2} (-x^2 + x + 2) dx$$

2c. Express the area  $A$  as integral(s) with respect to  $y$  (so you want  $dy$ ).

You do NOT have to evaluate the integral(s) nor do lots of algebra.

$$A = \int_{y=0}^{y=1} [(+\sqrt{y}) - (-\sqrt{y})] dy + \int_{y=1}^{y=4} [(+\sqrt{y}) - (y-2)] dy$$

$$\Leftrightarrow x = \pm \sqrt{y/3}$$

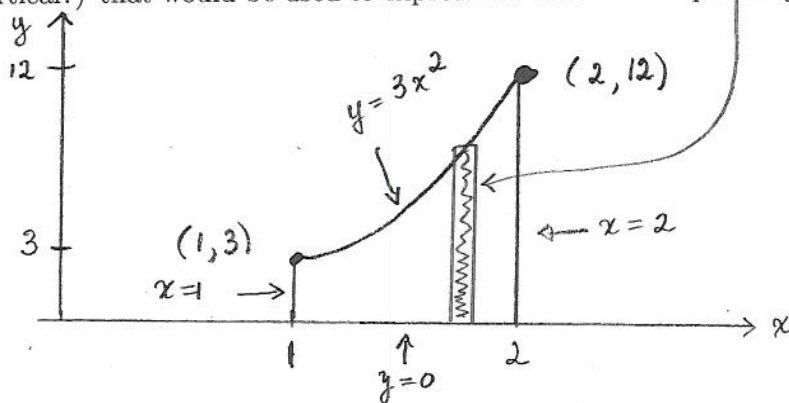
3. Sketched below is the region  $R$  that is enclosed by

$$y = 3x^2 \quad \text{and} \quad y = 0 \quad \text{and} \quad x = 1 \quad \text{and} \quad x = 2.$$

In each of problems 3a, 3b, 3c:

- $R$  will be revolved around some line to create a solid of revolution
- using either the disk, washer, or shell method, express the volume  $V$  of the resulting solid of revolution as one integral (and NOT as 2 or more integrals).
- In the space provided **below** each problem, make some *good enough sketch* (does not have to be too fancy) to indicate (i.e., help justify) your thinking/reasoning behind your solution
- you do not have to do lots of algebra to your integrand
- you do not have to integrate your integral.

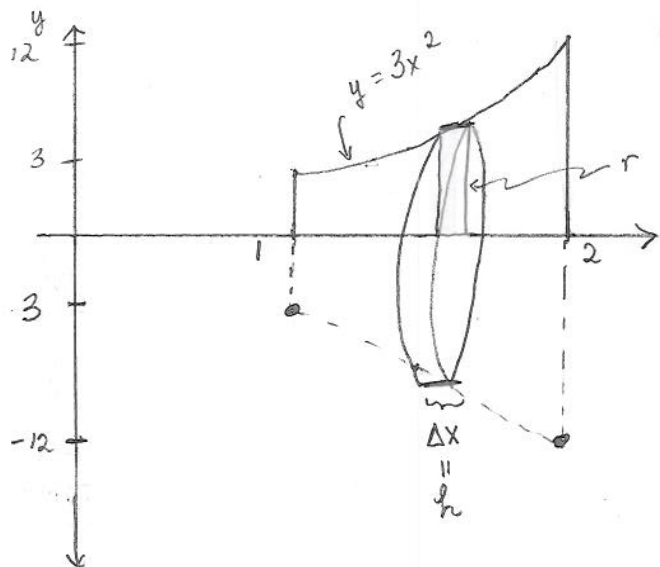
**Extra Credit/Hint** In the sketch below, draw in a typical rectangle (should it be horizontal or vertical?) that would be used to express the area of  $R$  as precisely 1 integral (and not 2 integrals).



3a. The volume  $V$  of the solid obtained by revolving the region  $R$  about the  $x$ -axis is

$$V = \int_{x=1}^{x=2} \pi (3x^2)^2 dx \quad \underline{\text{or}} \quad 9\pi \int_{x=1}^{x=2} x^4 dx$$

in terms of  $x$



Partition  $x$ -axis  $\Rightarrow$  everything  $\Rightarrow$  Disk Method

Volume of typical element

$$= (\text{area base})(\text{height})$$

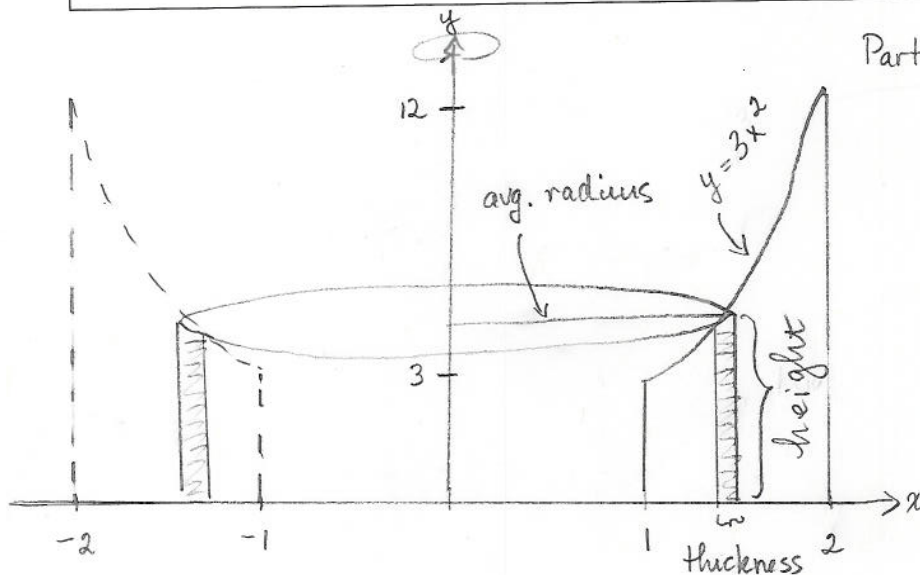
$$= (\pi r^2) h$$

$$= \pi (3x^2)^2 \Delta x$$

3b. The volume  $V$  of the solid obtained by revolving the region  $R$  about the  $y$ -axis is

$$V = \int_{x=1}^{x=2} 2\pi x (3x^2) dx \quad \text{or} \quad 6\pi \int_{x=1}^{x=2} x^3 dx$$

is in terms of  $x$

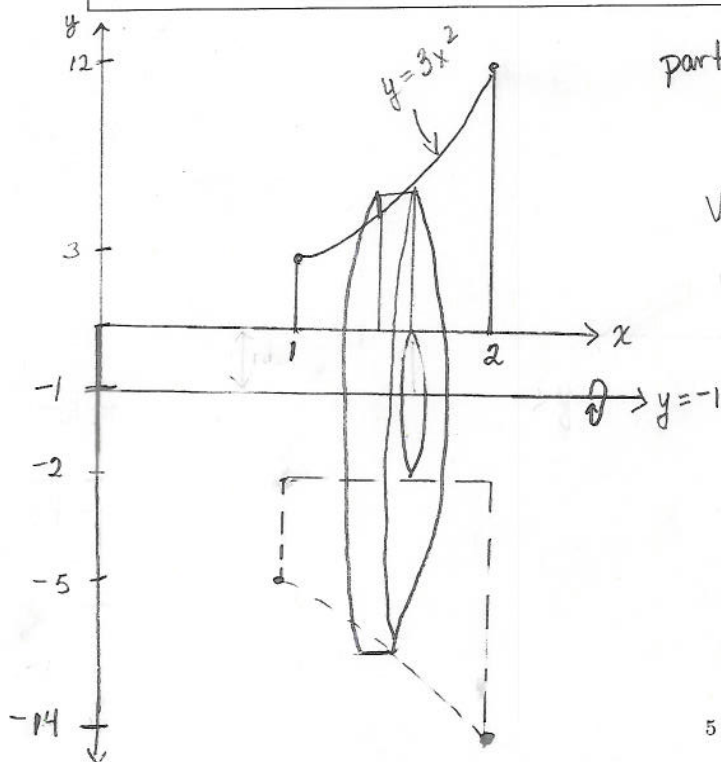


Partition  $x$ -axis  $\Rightarrow$  everything  $\Rightarrow$  Shell Method

$$\begin{aligned} \text{Volume of typical element} &= 2\pi (\text{avg. radius})(\text{height})(\text{thickness}) \\ &= 2\pi (x)(3x^2)(\Delta x) \end{aligned}$$

3c. The volume  $V$  of the solid obtained by revolving the region  $R$  about the line  $y = -1$  is

$$V = \int_{x=1}^{x=2} \pi [(3x^2+1)^2 - 1] dx \quad \text{or} \quad \pi \int_{x=1}^{x=2} (9x^4 + 6x^2) dx$$



partition  $x$ -axis  $\Rightarrow$  everything in terms of  $x$   
 $\Rightarrow$  Washer method.

$$\begin{aligned} \text{Volume of typical element} &= \text{Volume big} - \text{Volume little} \\ &= \pi (\text{radius}_{\text{big}})^2 (\text{height}) - \pi (\text{radius}_{\text{little}})^2 (\text{height}) \\ &= \pi (3x^2+1)^2 (\Delta x) - \pi (1)^2 (\Delta x) \\ &= \pi [(3x^2+1)^2 - 1] dx \end{aligned}$$

4. Express the arclength of the parameterized curve

$$x(t) = t^3$$

$$y(t) = 5t$$

from the point  $P = (1, 5)$  to the point  $Q = (8, 10)$  as an integral with respect to  $t$

$$\text{arclength} = \int_{t=1}^{t=2} \sqrt{(3t^2)^2 + (5)^2} dt \quad \text{or} \quad \int_{t=1}^{t=2} \sqrt{9t^4 + 25} dt$$

Once again, you do not have to do lots of algebra to your integrand nor integrate your integral.

$$P = (1, 5) \Leftrightarrow \begin{cases} 1 = t^3 \\ 5 = 5t \end{cases} \Leftrightarrow t = 1$$

$$Q = (8, 10) \Leftrightarrow \begin{cases} 8 = t^3 \\ 10 = 5t \end{cases} \Leftrightarrow t = 2$$

$$\frac{dx}{dt} = \frac{d}{dt} (t^3) = 3t^2$$

$$\frac{dy}{dt} = \frac{d}{dt} (5t) = 5$$

$$\text{arclength} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

5.

$$\int \frac{\sin(3x)}{17 + \cos(3x)} dx = -\frac{1}{3} \ln |17 + \cos(3x)| + C \quad \text{or} \quad -\frac{1}{3} \ln(17 + \cos(3x)) + C$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

$$u = 17 + \cos(3x)$$

$$du = -3 \sin(3x) dx$$

why don't we  
need the  
absolute value  
sign?

$$\int \frac{\sin(3x)}{17 + \cos(3x)} dx = \int \frac{1}{17 + \cos(3x)} [\sin(3x) dx]$$

$$= \int \frac{1}{u} \left[-\frac{1}{3} du\right]$$

$$= -\frac{1}{3} \int \frac{du}{u}$$

$$= -\frac{1}{3} \ln |u| + C$$

$$= -\frac{1}{3} \ln |17 + \cos 3x| + C$$

6.

$$\int \frac{e^x}{\sqrt{1 - e^{(2x)}}} dx = \sin^{-1}(e^x) + C$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

$$\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}$$

Recall  $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$

$$\int \frac{e^x dx}{\sqrt{1 - e^{(2x)}}} dx = \int \frac{1}{\sqrt{1 - (e^x)^2}} [e^x dx]$$

$$= \int \frac{1}{\sqrt{1 - u^2}} du$$

$$= \sin^{-1} \frac{u}{1} + C$$

$$= \sin^{-1}(e^x) + C$$

7. 
$$\int x^3 17^{(x^4)} dx = \frac{17^{(x^4)}}{4 \ln 17} + C$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

Recall 
$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$u = x^4$$

$$du = 4x^3 dx$$

$$\implies \frac{1}{4} du = x^3 dx$$

$$\begin{aligned} \int x^3 17^{(x^4)} dx &= \int 17^{(x^4)} [x^3 dx] \\ &= \int 17^u \left[ \frac{1}{4} du \right] \\ &= \frac{1}{4} \int 17^u du \\ &= \frac{1}{4} \frac{17^u}{\ln 17} + C \\ &= \frac{17^{(x^4)}}{4 \ln 17} + C \end{aligned}$$

8. Consider a right square pyramid, as sketched in (i) below, of height  $h$  where:

- the base of the pyramid is a square with side length  $l$
- the line segment connecting
  - the center  $C$  of the square base
  - the top point  $A$  of the pyramid (which is called the apex)
 forms a right angle with the square base and has length  $h$ .

Sketches (ii) and (iii) are for hints.

8a. In sketch (iii),  $C = (0, 0)$  and  $A = (0, h)$  and  $P = (\frac{l}{2}, 0)$

and the equation of the line  $L$  is  $y = \frac{-2h}{l}x + h$  or  $x = \frac{l}{2h}(h-y)$ .

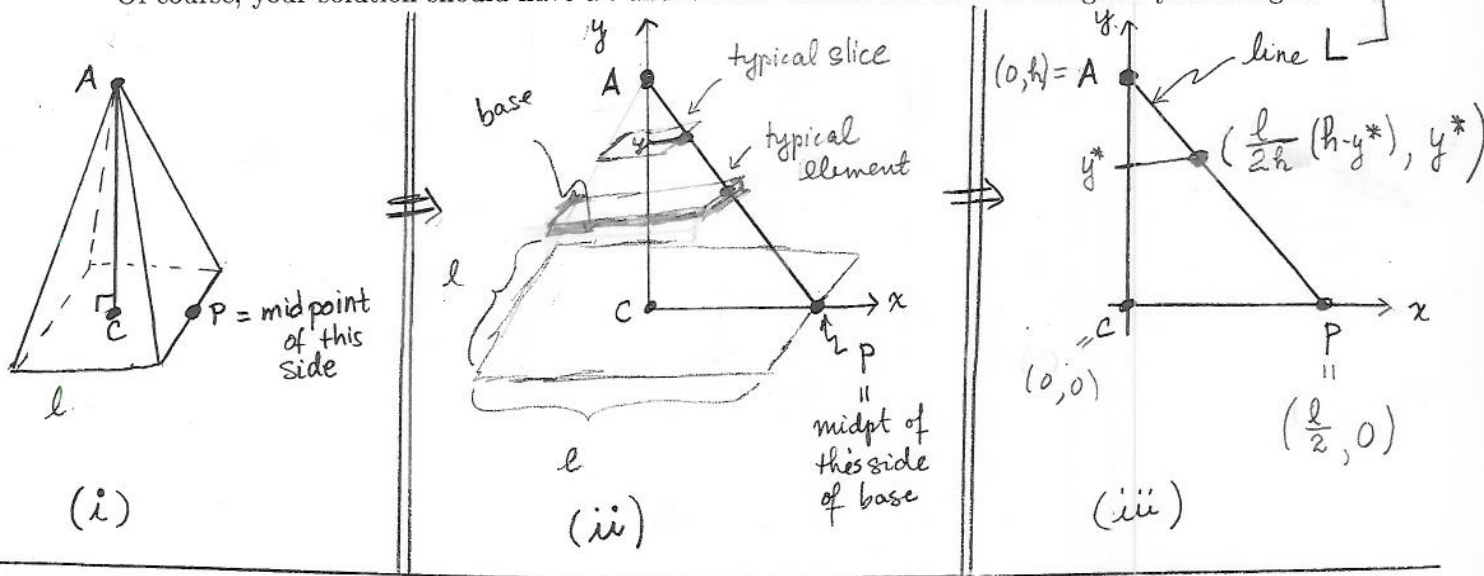
8b. Using the slicing method, express the volume of the pyramid as an integral.

$$\text{volume} = \int_{y=0}^{y=h} \frac{l^2}{h^2} (h-y)^2 dy$$

$$y = mx + b$$

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{0-h}{\frac{l}{2}-0} = -\frac{2h}{l}$$

Of course, your solution should have a  $l$  and  $h$  in it. You do not have to integrate your integral.



$$\begin{aligned} \text{Area of typical slice (at } y=y^*) &= [\text{length of side of square}]^2 = \left[ 2 \left( \frac{l}{2h} (h-y^*) \right) \right]^2 \\ &= \left[ \frac{l}{h} (h-y^*) \right]^2 = \frac{l^2}{h^2} (h-y^*)^2 \end{aligned}$$

$$\begin{aligned} \text{Volume of typical element (at } y=y^*) &= [\text{Area of typical slice (at } y=y^*)] [\text{height}] \\ &= \frac{l^2}{h^2} (h-y^*)^2 \Delta y \end{aligned}$$

$$\begin{aligned} \text{Volume} &= \int_{y=0}^{y=h} \frac{l^2}{h^2} (h-y)^2 dy \stackrel{\text{fun}}{=} \frac{l^2}{h^2} \int_{y=0}^{y=h} (y-h)^2 dy = \frac{l^2}{h^2} \left. \frac{(y-h)^3}{3} \right|_{y=0}^{y=h} \\ &= \frac{l^2}{3h^2} [0^3 - (-h)^3] = \frac{1}{3} l^2 h = \frac{1}{3} (\text{area base})(\text{height}) \end{aligned}$$