

| MARK BOX | | |
|-----------------------------|--------|--|
| PROBLEM | POINTS | |
| 1 a-y | 25 | |
| 2 | 10 | |
| 3 | 11 | |
| 4 | 11 | |
| 5 | 11 | |
| 6 | 11 | |
| 7 | 11 | |
| 8 take home | 10 | |
| 9 extra credit take home | 2 | |
| % | 100 | |

NAME: Solutions

please check the box of your section

 Section 005 (WF 8:00 am)

or

 Section 006 (WF 9:05 am)**INSTRUCTIONS:**

- (1) To receive credit you must:
 - (a) **work in a logical fashion, show all your work, indicate your reasoning;**
no credit will be given for an answer that just appears;
such explanations help with partial credit
 - (b) if a line/box is provided, then:
 - show you work BELOW the line/box
 - put your answer on/in the line/box
 - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
- (3) You may **not** use a calculator, books, personal notes.
- (4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- (5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8th ed.):
Sections 8.2, 8.3, 8.4, 8.5, 8.7, 8.8. .

Problem Inspiration:

1. You were warned.
2. An example from class.
3. Handout of 100 integrals # 7
4. Handout of 100 integrals # 23
5. Handout of 100 integrals # 47
6. Handout of 100 integrals # 26
7. Handout of 100 integrals - 4th page
8. just like the homework.
9. Section 8.8, number 16.

Hints:

- (1) You can check your answers to the indefinite integrals by differentiating.
- (2) For more partial credit, box your $u - du$ substitutions.

1. Fill in the blanks (each worth 1 point).

1a. $\int \frac{du}{u} = \ln |u| + C$

1b. If a is a constant and $a > 0$ but $a \neq 1$, then $\int a^u du = \frac{a^u}{\ln a} + C$

1c. $\int \cos u du = \sin u + C$

1d. $\int \sec^2 u du = \tan u + C$

1e. $\int \sec u \tan u du = \sec u + C$

1f. $\int \sin u du = -\cos u + C$

1g. $\int \csc^2 u du = -\cot u + C$

1h. $\int \csc u \cot u du = -\csc u + C$

1i. $\int \tan u du = -\ln |\cos u| + C$ ~~or $\ln |\sec u| + C$~~ ~~$+C$~~

1j. $\int \cot u du = \ln |\sin u| + C$ ~~or $-\ln |\csc u| + C$~~ ~~$+C$~~

1k. $\int \sec u du = \ln |\sec u + \tan u| + C$ ~~or $-\ln |\sec u - \tan u| + C$~~ ~~$+C$~~

1l. $\int \csc u du = -\ln |\csc u + \cot u| + C$ ~~or $\ln |\csc u - \cot u| + C$~~ ~~$+C$~~

1m. If a is a constant and $a > 0$ then $\int \frac{1}{\sqrt{a^2 - u^2}} du = \arcsin \frac{u}{a} + C$

1n. If a is a constant and $a > 0$ then $\int \frac{1}{a^2 + u^2} du = \frac{1}{a} \arctan \frac{u}{a} + C$

1o. If a is a constant and $a > 0$ then $\int \frac{1}{u\sqrt{u^2 - a^2}} du = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

1p. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polynomials

and $[\text{degree of } f] \geq [\text{degree of } g]$, then one must first do long division

1q. Integration by parts formula: $\int u dv = uv - \int v du$

1r. Trig substitution: (recall that the *integrand* is the function you are integrating)
if the integrand involves $a^2 - u^2$, then one makes the substitution $u = a \sin \theta$

1s. Trig substitution:
if the integrand involves $a^2 + u^2$, then one makes the substitution $u = a \tan \theta$

1t. Trig substitution:
if the integrand involves $u^2 - a^2$, then one makes the substitution $u = a \sec \theta$

1u. trig formula ... your answer should involve trig functions of θ , and not of 2θ : $\sin(2\theta) = 2 \sin \theta \cos \theta$

1v. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$

1w. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$

1x. trig formula ... since $\cos^2 \theta + \sin^2 \theta = 1$, we know that the corresponding relationship between tangent (i.e., \tan) and secant (i.e., \sec) is $1 + \tan^2 \theta = \sec^2 \theta$

1y. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ RADIANS. (your answer should be an angle)

2.

$$\int_{x=17}^{x=\infty} \frac{1}{x^2} dx = \frac{1}{17}$$

$$\int_{x=17}^{x=\infty} x^{-2} dx = \lim_{b \rightarrow \infty} \int_{x=17}^{x=b} x^{-2} dx$$

$$= \lim_{b \rightarrow \infty} \left. \frac{x^{-1}}{-1} \right|_{x=17}^{x=b}$$

$$= \lim_{b \rightarrow \infty} \left. -\frac{1}{x} \right|_{x=17}^{x=b}$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{1}{b} - \left(-\frac{1}{17} \right) \right]$$

$$= \lim_{b \rightarrow \infty} -\frac{1}{b} + \lim_{b \rightarrow \infty} \frac{1}{17}$$

$$= 0 + \frac{1}{17}$$

3.

3a. $D_x \tan x = \sec^2 x$

3b. $D_x \sec x = \sec x \tan x$

3c. $\int \tan^2 x \, dx = \tan x - x + C$

Hint: problems 1x, 3a, and 3b might come in handy for problem 3c.

$$\int \tan^2 x \, dx \stackrel{(1x)}{=} \int (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \, dx - \int dx$$

\downarrow (3a)

$$= \tan x - x + C$$

4.

$$\int \ln(3x+6) dx = (x+2) \ln(3x+6) - x + C$$

Hint: be clever in your choice of v to avoid unneeded work.

Note that $\frac{3}{3x+6} = \frac{3}{3(x+2)} = \frac{1}{x+2}$.

Parts - Lesson #4 $f(x) = \ln(3x+6)$ is easy to differentiate but hard to integrate so try letting $u = f(x)$.

Way #1

$$u = \ln(3x+6)$$

$$dv = dx$$

$$du = \frac{3}{3x+6} dx$$

$$v = x$$

$$3x+6 \frac{1}{3x+6}$$

$$\frac{3x+6}{3x+6}$$

$$-6$$

$$\int \ln(3x+6) dx = x \ln(3x+6) - \int \frac{3x}{3x+6} dx$$

$$= x \ln(3x+6) - \int \left(1 - \frac{6}{3x+6}\right) dx$$

$$= x \ln(3x+6) - \int dx + 6 \frac{1}{3} \int \frac{3 dx}{3x+6}$$

$$= x \ln(3x+6) - x + 2 \ln(3x+6) + C$$

Way #2

$$u = \ln(3x+6)$$

$$dv = dx$$

$$du = \frac{3}{3x+6} dx = \frac{1}{x+2} dx$$

$$v = x+2$$

$$\int \ln(3x+6) dx = (x+2) \ln(3x+6) - \int (x+2) \cdot \left(\frac{1}{x+2}\right) dx$$

$$= (x+2) \ln(3x+6) - \int dx$$

$$= (x+2) \ln(3x+6) - x + C$$

5.

$$\int \frac{x^4 + 2x + 2}{x^5 + x^4} dx = -\frac{2}{3x^3} + \ln|x+1| + C$$

Hint: $x^5 + x^4 = (x^4)(x+1) = (x-0)^4(x+1)^1$.

$$\frac{x^4 + 2x + 2}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1} \quad \leftarrow \text{Note } x^4 = (x-0)^4$$

$$\Rightarrow \frac{x^4 + 2x + 2}{x^4(x+1)} = \frac{Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4}{x^4(x+1)}$$

$$\Rightarrow x^4 + 2x + 2 = Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4$$

$$x=0 \Rightarrow 2 = D$$

$$x=-1 \Rightarrow 1 = E$$

$$x^4 : 1 = A + E \rightarrow A = 0$$

$$x^3 : 0 = A + B$$

$$x^2 : 0 = B + C$$

$$x : 2 = C + D$$

$$\text{constant} : 2 = D$$

$$C = 0 \rightarrow B = 0$$

$$\int \frac{x^4 + 2x + 2}{x^5 + x^4} dx = \int \left(\frac{2}{x^4} + \frac{1}{x+1} \right) dx$$

$$= \int 2x^{-4} dx + \int \frac{dx}{x+1} = \frac{2x^{-3}}{-3} + \ln|x+1| + c$$

6.

$$\int \frac{x^2}{\sqrt{16-x^2}} dx = 8 \sin^{-1} \left(\frac{x}{4} \right) - \frac{x \sqrt{16-x^2}}{2} + C$$

Hint: problems 1 u - w might come in handy.

$$x = 4 \sin \theta \Rightarrow \sin \theta = \frac{x}{4} \Rightarrow \begin{array}{c} 4 \\ \theta \\ \sqrt{16-x^2} \end{array} \Rightarrow \cos \theta = \frac{\sqrt{16-x^2}}{4}$$

$$dx = 4 \cos \theta d\theta$$

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16(1-\sin^2\theta)} = 4\sqrt{\cos^2\theta} = 4\cos\theta$$

$$\int \frac{x^2}{\sqrt{16-x^2}} dx = \int \frac{(16\sin^2\theta)(4\cos\theta d\theta)}{4\cos\theta} = 16 \int \sin^2\theta d\theta$$

$$\stackrel{(1w)}{=} \frac{16}{2} \int (1 - \cos 2\theta) d\theta = 8 \left[\theta - \frac{\sin(2\theta)}{2} \right] + C$$

$$\stackrel{(1w)}{=} 8\theta - 4\sin(2\theta) + C = 8\theta - 4 \cdot 2 \sin\theta \cos\theta$$

$$= 8 \sin^{-1} \left(\frac{x}{4} \right) - 8 \left(\frac{x}{4} \right) \left(\frac{\sqrt{16-x^2}}{4} \right) + C$$

7.

$$\int \csc^3 x \, dx = \frac{1}{2} \left(-\csc x \cot x + \ln |\csc x - \cot x| \right) + C$$

Hint: bring to the other side idea, similar to how to do $\int \sec^3 x \, dx$. Helpful:

- $D_x \cot x = -\csc^2 x$
- $D_x \csc x = -\csc x \cot x$
- $\cot^2 x + 1 = \csc^2 x$

$$u = \csc x \quad dv = \csc^2 x \, dx$$

$$du = -\csc x \cot x \, dx \quad v = -\cot x$$

$$\begin{aligned} \int \csc^3 x \, dx &= -\csc x \cot x - \int \csc x \cot^2 x \, dx \\ &= -\csc x \cot x - \int \csc x (\csc^2 x - 1) \, dx \\ &= -\csc x \cot x - \int \csc^3 x \, dx + \int \csc x \, dx \end{aligned}$$

\Rightarrow

$$\begin{aligned} 2 \int \csc^3 x \, dx &= -\csc x \cot x + \int \csc x \, dx \\ &= -\csc x \cot x - \ln |\csc x + \cot x| + C \\ &\stackrel{\text{or}}{=} -\csc x \cot x + \ln |\csc x - \cot x| + C \end{aligned}$$

8. Numerical Integration . Let

$$I = \int_{x=1}^{x=3} \frac{1}{x^2} dx .$$

The 10 steps of this problem are similar to the homework but the number of subintervals is 6 and not 10. On the parts that say "Do not use a calculator", you need not do alot of arithmetic.

8-1. Find the exact value of I , without using a calculator. Your answer should be a fraction, without decimal places.

$$I = \frac{2}{3}$$

$$\int_{x=1}^{x=3} x^{-2} dx = \frac{x^{-1}}{-1} \Big|_{x=1}^{x=3} = \frac{1}{x} \Big|_{x=1}^{x=3} = 1 - \frac{1}{3} = \frac{2}{3}$$

8-2. Find an approximation for I , using part 1 and your calculator, to as many decimal places as your calculator will give you.

$$I \approx .6666666666666666\bar{6}$$

8-3. Approximate I using the Trapezoid Rule T_n with $n = 6$ subintervals. Do not use a calculator (so your answer will have several numbers added together).

or $\frac{1}{6}$ →

$$T_6 = \frac{3-1}{2(6)} \left[1(1) + 2\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{5}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{3}{7}\right)^2 + 2\left(\frac{3}{8}\right)^2 + 1\left(\frac{1}{3}\right)^2 \right]$$

Divide $[1, 3]$ into 6 subintervals so $\Delta x = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$

| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | x_6 |
|-------------------|---------------|---------------|-----------------|---------------|---------------|-------------------|
| $a=1=\frac{3}{3}$ | $\frac{4}{3}$ | $\frac{5}{3}$ | $\frac{6}{3}=2$ | $\frac{7}{3}$ | $\frac{8}{3}$ | $\frac{9}{3}=3=b$ |

8-4. Find an approximation for T_6 , using part 3 and your calculator, to as many decimal places as your calculator will give you.

$$T_6 \approx .6841180083$$

8-5. Approximate I using Simpson's Rule S_{2n} with $2n = 6$ subintervals. Do not use a calculator (so your answer will have several numbers added together).

$$S_6 = \frac{1}{3} \left(\frac{3-1}{6} \right) \left[1 + 4\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{5}\right)^2 + 4\left(\frac{1}{2}\right)^2 + 2\left(\frac{3}{7}\right)^2 + 4\left(\frac{3}{8}\right)^2 + 1\left(\frac{1}{3}\right)^2 \right]$$

↓

or $\frac{1}{9}$ or $\frac{1}{3} \cdot \frac{1}{3}$ or ...

8-6. Find an approximation for S_6 , using part 5 and your calculator, to as many decimal places as your calculator will give you.

$$S_6 \approx .6678842278$$

8-*. Find the first 4 derivatives of $f(x) = x^{-2}$.

$$\begin{aligned} f(x) &= x^{-2} &&= x^{-2} \\ f'(x) &= -2x^{-3} &&= -2! x^{-3} \\ f^{(2)}(x) &= 6x^{-4} &&= 3! x^{-4} \\ f^{(3)}(x) &= -24x^{-5} &&= -4! x^{-5} \\ f^{(4)}(x) &= 120x^{-6} &&= 5! x^{-6} \end{aligned}$$

8-7. Use inequality (11), page 563, to find an upper bound on the error in part 3. Do not use a calculator.

$$|T_6 - I| \leq \frac{(3-1)^3}{12 \cdot 6^2} (6) \underline{\underline{=}} \underline{\underline{\frac{1}{9}}}$$

$$K_2 = \max_{a \leq x \leq b} |f''(x)| = \max_{1 \leq x \leq 3} |6x^{-4}| = \max_{1 \leq x \leq 3} \frac{6}{x^4}$$
$$= \frac{6}{1^4} = 6$$

8-8. Use your calculator to approximate the error estimate in part 7 to as many decimal places as your calculator will give you.

$$|T_6 - I| \approx 0.1$$

8-9. Use inequality (12), page 563 to find an upper bound on the error in part 5. Do not use a calculator.

$$|S_6 - I| \leq \frac{(3-1)^5 (120)}{180 (6)^4} \underline{\underline{=}} \underline{\underline{\frac{2^2}{3^5}}} \underline{\underline{=}} \underline{\underline{\frac{4}{243}}}$$

$$K_4 = \max_{a \leq x \leq b} |f^{(4)}(x)| = \max_{1 \leq x \leq 3} \left| \frac{120}{x^6} \right| = \max_{1 \leq x \leq 3} \frac{120}{x} = 120$$

8-10 Use your calculator to approximate the error estimate in part 9, to as many decimal places as your calculator will give you.

$$|S_6 - I| \approx 0.0164609053$$

9. Extra Credit - number 16 from section 8.8.

$$\int_{x=-\infty}^{x=\infty} \frac{e^{-x}}{1+e^{-2x}} dx = \frac{\pi}{2}$$

First $u = e^{-x}$
 $du = -e^{-x} dx$

$$\int \frac{e^{-x}}{1+e^{-2x}} dx = - \int \frac{-e^{-x} dx}{1+(e^{-x})^2} = - \int \frac{du}{1+u^2} = -\arctan u + C$$

$$= -\arctan e^{-x} + C$$

$$\int_{x=-\infty}^{x=0} \frac{e^{-x}}{1+e^{-2x}} dx = \lim_{a \rightarrow -\infty} \int_{x=a}^{x=0} \frac{e^{-x} dx}{1+e^{-2x}} = \lim_{a \rightarrow -\infty} -\arctan e^{-x} \Big|_{x=a}^{x=0}$$

$$= \lim_{a \rightarrow -\infty} (-\arctan e^0 - -\arctan e^{-a})$$

$$= \lim_{a \rightarrow -\infty} (\arctan 1 + \arctan e^{-a})$$

$$= \lim_{a \rightarrow -\infty} \left(-\frac{\pi}{4} + \arctan e^{-a}\right)$$

$$= -\frac{\pi}{4} + \lim_{a \rightarrow -\infty} \arctan e^{-a}$$

↳ think of as

this much is not math. correct - idea behind it all.

$$\begin{aligned} \arctan e^{-\infty} &= \arctan e^{\infty} \\ &= \arctan \infty \\ &= \frac{\pi}{2} \end{aligned}$$

$$= -\frac{\pi}{4} + \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$\int_{x=0}^{x=\infty} \frac{e^{-x}}{1+e^{-2x}} dx = \lim_{b \rightarrow \infty} \int_{x=0}^{x=b} \frac{e^{-x} dx}{1+e^{-2x}} = \lim_{b \rightarrow \infty} -\arctan e^{-x} \Big|_{x=0}^{x=b}$$

$$= \lim_{b \rightarrow \infty} \left(-\arctan e^{-b} - -\arctan e^0 \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\arctan e^{-b} + \arctan 1 \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\arctan e^{-b} + \frac{\pi}{4} \right)$$

$$= \frac{\pi}{4} + \lim_{b \rightarrow \infty} -\arctan e^{-b}$$

(not math. correct but do you get the idea?)

↳ think of as $-\arctan e^{-\infty} = -\arctan 0 = -0 = 0$

$$= \frac{\pi}{4} + 0.$$

$$\int_{x=-\infty}^{x=\infty} \frac{e^{-x}}{1+e^{-2x}} dx = \int_{x=-\infty}^{x=0} \frac{e^{-x}}{1+e^{-2x}} dx + \int_{x=0}^{x=+\infty} \frac{e^{-x}}{1+e^{-2x}} dx$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$