Prof. Girardi	Math 142.003/004	Fall 2005	12.08.05	Final Exam

MARK BOX		
PROBLEM	POINTS	
1 a – r	35	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
%	100	

NAME: Answer Key

SSN: ______________________

please check the box of your section below

☐ **Section 003 (MW 9:05 pm)**

or

☐ **Section 004 (MW 10:10 pm)**

INSTRUCTIONS:

(1) To receive credit you must:
   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
   (b) when applicable put your answer on/in the line/box provided
   (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): Sections 6.6, 6.8, 6.9, 7.1–7.4, 7.6, 7.7, 8.1–8.8, 10.1–10.10. .

Problem Inspiration: homework and old exams.

Solutions will be available on the course homepage later this afternoon.

1. Fill in the blanks (each worth 1 point) and boxes (each worth 2 points).

1a. $\int \frac{dx}{x}$ = ln $|x| + C$

1b. $D_x\, e^x$ = e^x

1c. If $a > 0$ but $a \neq 1$, then $D_x\, a^x$ = $a^x \ln a$

Hint: $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$. Your answer should **not** have an "e" in it.

$$\boxed{\begin{aligned} &D_x\,(a^x) \\ &= D_x\, e^{x \ln a} \\ &= \left(e^{x\ln a}\right)\left(D_x\, e^{x\ln a}\right) \\ &= (a^x)\,(\ln a) \end{aligned}}$$

1d. $D_x\, \tan x$ = $\sec^2 x$

1e. $\int \sec x\ \tan x$ = $\sec x$ $+C$

1f. Integration by parts formula:

$\int$ ~~$u\,dx$~~ $u\,dv$ = $uv - \int v\,du$

1g. Trig substitution: (recall that the *integrand* is the function you are integrating)

if the integrand involves a^2-u^2, then one makes the substitution u = $a \sin\theta$

1h. Trig substitution:

if the integrand involves a^2+u^2, then one makes the substitution u = $a \tan\theta$

1i. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do long division

1j. Integral Test: $a_n > 0$

Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a continuous function
- f is a positive function
- f is a decreasing function .

Then $\sum a_n$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

1k. Comparison Test: $a_n > 0$

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges .
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges .

1l. Limit Comparison Test: $a_n > 0$

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges .

1m. Ratio Test: $a_n > 0$

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

- If $\rho <$ ___1___ then $\sum a_n$ converges.
- If $\rho >$ ___1___ then $\sum a_n$ diverges.
- If $\rho =$ ___1___ then the test is inconclusive.

1n. Root Test: $a_n > 0$

Let $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If $\rho <$ ___1___ then $\sum a_n$ converges.
- If $\rho >$ ___1___ then $\sum a_n$ diverges.
- If $\rho =$ ___1___ then the test is inconclusive.

1o. Alternating Series Test: $a_n > 0$

If

- a_n ___>___ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ ___0___

then $\sum(-1)^n a_n$ ___converges___

1p. n^{th}-term test: a_n's are arbitrary

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ ___diverges___.

1q. Consider the interval $I = (a - R, a + R)$ center about $x = a$ and of radius R.

Let $y = f(x)$ be a function that can be differentiated N times $x = a$. Then the N^{th}-order Taylor polynomial $y = P_N(x)$ of f about a is (your answer should have a summation sign $\sum$ in it)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

1r. Consider the interval $I = (a - R, a + R)$ center about $x = a$ and of radius R.

Let $y = f(x)$ be a function that can be differentieated $N + 1$ times for each $x \in I$.

Consider the the N^{th}-order Taylor Reminder term $R_N(x)$, where $f(x) = P_N(x) + R_N(x)$.

Then an upper bound for $|R_N(x)|$ for an $x \in I$ is:

$$|R_N(x)| \leq \left[\max_{x \in (a-R, a+R)} |f^{(N+1)}(x)| \right] \frac{1}{(N+1)!} |x-a|^{N+1}$$

2. $D_x\ (\cos(\ln x)) = \left[-\sin(\ln x)\right] \cdot \frac{1}{x}$

3. $D_x\ 7^{(x^2)} = 7^{(x^2)}\ (2x)\ (\ln 7)$

4. $$\boxed{\int (\tan x)\ (\sec^7 x)\ dx \ = \ \frac{\sec^7 x}{7} + C}$$

Remark: box your substitution box for more partial credit.

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x \tan x \, dx \end{aligned}}$$

$$\int (\tan x)\,(\sec^7 x)\,dx = \int \sec^6 x \,(\sec x \tan x \, dx)$$

$$= \int u^6 \, du$$

$$= \frac{u^7}{7} + C$$

$$= \frac{\sec^7 x}{7} + C$$

5. $$\boxed{\int x^2 \arctan x \, dx = \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{\ln|1+x^2|}{6} + C}$$

Remark: box your substitution box for more partial credit.

parts

$$\boxed{\begin{array}{ll} u = \arctan x & dv = x^2\,dx \\ du = \dfrac{dx}{1+x^2} & v = \dfrac{x^3}{3} \end{array}}$$

Long Division

$$\begin{array}{r|l} & x \\ x^2+1 & x^3 \\ & x^3 + x \\ \hline & -x \end{array}$$

$$\int x^2 \tan^{-1} x \, dx \overset{\text{parts}}{=} \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{x^3}{1+x^2} \, dx$$

$$\overset{\text{L.D.}}{=} \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \left[x - \frac{x}{x^2+1} \right] dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int x\,dx + \frac{1}{3}\frac{1}{2} \int \frac{1}{x^2+1} (2x\,dx)$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6} \ln|x^2+1| + C$$

6. $$\boxed{\int \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$$

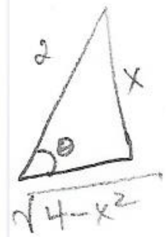
Remark: box your substitution box for more partial credit.

$$a^2 - u^2 \Rightarrow u = a\sin\theta$$

$$x = 2\sin\theta \longrightarrow \sin\theta = \frac{x}{2}$$

$$dx = 2\cos\theta\, d\theta$$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta \longrightarrow \cos\theta = \frac{\sqrt{4-x^2}}{2}$$



$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta\, 2\cos\theta\, d\theta}{2\cos\theta} = 4\int \sin^2\theta\, d\theta$$

$$= 4 \cdot \frac{1}{2} \int (1-\cos 2\theta)\, d\theta$$

$$= 2\theta - 2 \cdot \frac{\sin 2\theta}{2} + C = 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

7. Let R be the region enclosed by

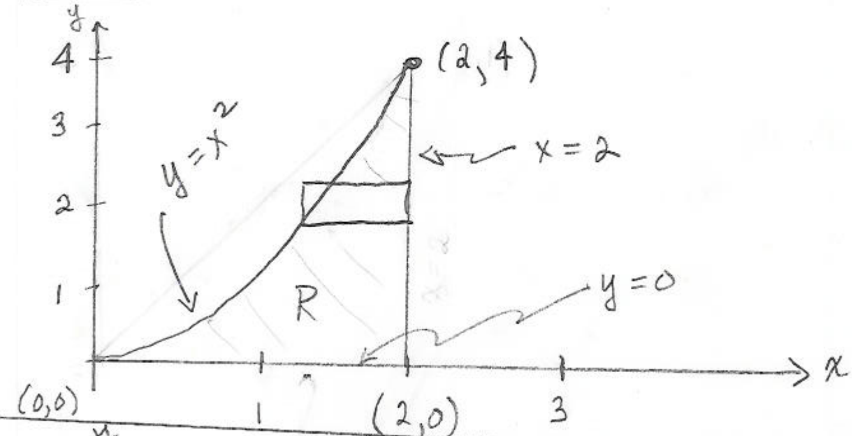
$$y = x^2 \qquad \text{and} \qquad x = 2 \qquad \text{and} \qquad y = 0\,.$$

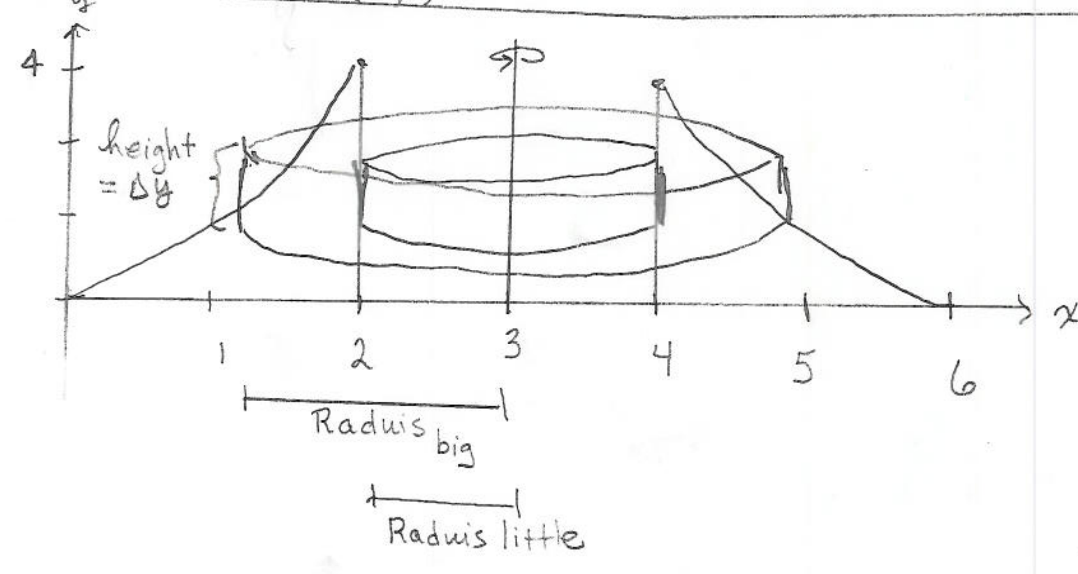
Let V be the volume of the solid obtained by revolving the region R about the line $x = 3$.

7a. Make a rough sketch below of the region R, labeling the important points.

7b. Using the ~~disk~~/washer method, express the volume V as an integral (or maybe 2 integrals). You do NOT have to evaluate the integral(s).

$$\boxed{V = \pi \int_{y=0}^{y=4} \left[(3-\sqrt{y})^2 - 1 \right] dy}$$





Volume of typical element $= \pi R_{big}\,(\text{height}) - \pi R_{litte}\,(\text{height})$

$$= \pi\,(R_{big} - R_{little})\,(\text{height}) = \pi\left[(3-\sqrt{y})^2 - (3-2)^2\right] dy$$

8. $$\lim_{n\to\infty} \frac{3n^{5/2}+7n^2+9}{-17n^{5/2}+3n^2-9n-18} = \frac{3}{-17}$$

divide thru by $n^{\text{highest power}} = n^{5/2}$

$$\lim_{n\to\infty} \frac{3n^{5/2}+7n^2+9}{-17n^{5/2}+3n^2-9n-18} = \lim_{n\to\infty} \frac{\frac{3n^{5/2}}{n^{5/2}}+\frac{7n^{4/2}}{n^{5/2}}+\frac{9}{n^{5/2}}}{\frac{-17n^{5/2}}{n^{5/2}}+\frac{3n^{4/2}}{n^{5/2}}-\frac{9n^{2/2}}{n^{5/2}}-\frac{18}{n^{5/2}}}$$

$$= \lim_{n\to\infty} \frac{3+\frac{7}{n^{1/2}}+\frac{9}{n^{5/2}}}{-17+\frac{3}{n^{1/2}}-\frac{9}{n^{3/2}}-\frac{18}{n^{5/2}}} = \frac{3+0+0}{-17+0-0-0}$$

9. $$\lim_{n\to\infty} \frac{8n^{15}-7n^{10}+19}{-5n^{13}+6n^8-6n^5+9} = -\infty$$

divide thru by $n^{\text{highest power}} = n^{15}$

$$\lim_{n\to\infty} \frac{8n^{15}-7n^{10}+19}{-5n^{13}+6n^8-6n^5+9} = \lim_{n\to\infty} \frac{8-\frac{7}{n^5}+\frac{19}{n^{15}}}{-\frac{5}{n^2}+\frac{6}{n^7}-\frac{6}{n^{10}}+\frac{9}{n^{15}}}$$

$$= \cancel{\frac{8}{0}} = \pm\infty \quad (?)$$

$$(?)\quad \frac{8n^{15}-7n^{10}+19}{-5n^{13}+6n^8-6n^5+9} \overset{n \text{ big}}{\approx} \frac{8n^{15}}{-5n^{13}} = \frac{-8}{5}n^2 < 0.$$

On problems 10 and 11b, check the correct box and then indicte your reasoning below. A correctly checked box without appropriate explanation will receive no points.

10. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$

- [ ] absolutely convergent
- [x] conditionally convergent
- [ ] divergent

abs. conv? Consider $\sum \frac{\ln n}{n}$

CT. $\frac{1}{n} \leq \frac{\ln n}{n}$ for n big enough

$\sum \frac{1}{n}$ divg (harmonic series)(p-series, $p=1\leq 1$)

CT $\Rightarrow \sum \frac{\ln n}{n}$ divg. so not abs. conv.

Cond. conv? try AST so $a_n = \frac{\ln n}{n}$

① $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ $x \geq 3$

so $a_n > a_{n+1}$ for $n \geq 3$

② $\lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

AST $\Rightarrow \sum (-1)^n \frac{\ln n}{n}$ conv.

11a Let $a_n = \frac{n^3\ (n!)}{(2n)!}$ Find $\frac{a_{n+1}}{a_n}$. Simplify your answer so that no factorial sign (i.e., !) appears.

answer: $\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^3 \frac{n+1}{(2n+1)(2n+2)} \overset{\text{or}}{=} \left(\frac{n+1}{n}\right)^3 \frac{1}{2(2n+1)}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3\ ((n+1)!)}{(2(n+1))!} \cdot \frac{(2n)!}{n^3 \cdot n!} = \frac{(n+1)^3}{n^3} \cdot \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!}$$

$$= \left(\frac{n+1}{n}\right)^3 \frac{n!\ (n+1)}{n!} \frac{(2n)!}{(2n)!\ (2n+1)\ (2n+2)}$$

11b. $\sum_{n=17}^{\infty} (-1)^n \frac{n^3\ (n!)}{(2n)!}$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

abs conv? Consider $\sum \frac{n^3\ n!}{(2n)!}$ & use Ratio Test

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{2(2n+1)}$$

$$= 1^3 \cdot 0 = 0 < 1$$

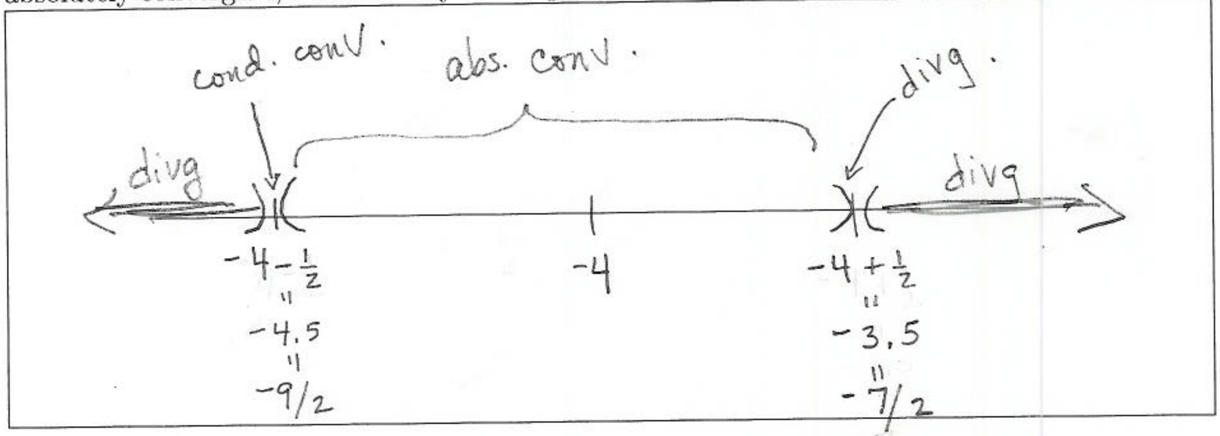
Ratio Test $\Rightarrow \sum \frac{n^3\ n!}{(2n)!}$ conv.

center $= -4$

12. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+8)^n}{n}. \quad = \sum_{n=1}^{\infty} \frac{2^n (x - {-4})^n}{n}$$

As we did in class, in the box below draw a diagram indicating for which x's this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning.



Ratio Test for abs. conv

$$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1} (x+4)^{n+1}}{n+1} \cdot \frac{n}{2^n (x+4)^n} \right| = 2\,|x+4| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 2\,|x - {-4}| < 1 \iff |x - {-4}| < \tfrac{1}{2}$$

Check endpoints

$x = -7/2$: $\sum \frac{(2x+8)^n}{n} = \sum \frac{1}{n}$ ← divges, harmonic series or p-series, $p = 1 \leq 1$

$x = -9/2$ $\sum \frac{(2x+8)^n}{n} = \sum \frac{(-1)^n}{n}$ ← cond. conv.

- $\sum \frac{1}{n}$ divg
- $\sum \frac{(-1)^n}{n}$ conv by AST

13. Let

$$f(x) = (1+x)^{3/2}$$

and $a = 0$.

Find the 3rd-order Taylor polynomial of $y = f(x)$ about (or at) $a = 0$.

$P_3(x) =$

$f(x) = (1+x)^{3/2}$ $\qquad$ $f(0) = 1$

$f'(x) = \frac{3}{2}(1+x)^{\frac{1}{2}}$ $\qquad$ $f'(0) = \frac{3}{2}$

$f^{(2)}(x) = \frac{3}{2} \cdot \frac{1}{2} (1+x)^{-1/2}$ $\qquad$ $f^{(2)}(0) = \frac{3}{4}$

$f^{(3)}(x) = -\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} (1+x)^{-3/2}$ $\qquad$ $f^{(3)}(0) = -\frac{3}{8}$

$$P_3(x) = \sum_{n=0}^{3} \frac{f^{n}(a)}{n!} (x-a)^n$$

$\frac{3}{8 \cdot 2 \cdot 3} = \frac{1}{16}$

$$= \boxed{1 + \frac{3}{2}x + \frac{3}{4} \cdot \frac{1}{2!} x^2 + -\frac{3}{8} \frac{1}{3!} x^3}$$

$$= \boxed{1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3}$$

either one is ok

14. As in problem 13, let

$$f(x) = (1+x)^{3/2}$$

and $a = 0$.

Let $f(x) = P_3(x) + R_3(x)$, where $y = P_3(x)$ is the 3rd-order Taylor polynomial of $y = f(x)$ about $a = 0$ and $y = R_3(x)$ is the corresponding remainder (i.e., error) term.

Consider the interval $I = (-0.5, 0.5)$ center about $a = 0$. Fix an $x \in I$. Find a good upper bound for $|R_3(x)|$.

$$\boxed{|R_3(x)| \le \frac{9}{16} \quad \frac{1}{(0.5)^{5/2}} \quad \frac{|x|^4}{4!}}$$

Remark: you only have to carry out the algebra as far as I indicated in class.

$$f^{(4)}(x) = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(1+x)^{-5/2} = \frac{9}{2^4}(1+x)^{-5/2} \underset{\text{or}}{=} \frac{9}{16}(1+x)^{-5/2}$$

$$|R_3(x)| \le \left[\max_{x\in(-\frac{1}{2},\frac{1}{2})} |f^4(x)|\right] \frac{1}{4!}|x-0|^4$$

$$= \left[\max_{x\in(-\frac{1}{2},\frac{1}{2})} \frac{9}{16}\cdot\frac{1}{|1+x|^{5/2}}\right]\frac{|x|^4}{4!}$$

$$\boxed{\begin{array}{ll} \text{if } -\frac{1}{2} < x < \frac{1}{2} & \text{then } \frac{1}{2} < 1+x < \frac{3}{2} \\ & \text{so } \frac{1}{2} < |1+x| < \frac{3}{2} \end{array}}$$

$$\le \frac{9}{16}\cdot\frac{1}{(1/2)^{5/2}}\cdot\frac{|x|^4}{4!}$$ ok to her

$$= \frac{9}{2^4}\, 2^2\cdot 2^{1/2}\cdot\frac{|x|^4}{4!} = \frac{9\sqrt{2}}{4}\cdot\frac{|x|^4}{4!}$$