MARK BOX		
PROBLEM	POINTS	
1	36	
2	6	
3	2	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
%	100	

NAME: Answer Key

SSN: ______________________

please check the box of your section below

☐ **Section 003 (MW 9:05 pm)**

or

☐ **Section 004 (MW 10:10 pm)**

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) when applicable put your answer on/in the line/box provided
  (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): Sections 10.1 – 10.6 .

Problem Inspiration:

1-3. a course handout - you were warned
4-6. homework from § 10.1
7. homework from § 10.3
8-10. homework from § 10.6

Solutions will be available on the course homepage later this afternoon.

For problems 1, 2, and 3, fill in the blanks.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1. For problem 1, let $\sum a_n$ be a **positive-termed** series (i.e. $a_n \geq 0$ for each $n \in \mathbb{N}$).

1a. Integral Test Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a continuous function
- f is a positive function
- f is a decreasing function .

Then $\sum a_n$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

1b. Comparison Test

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converge, then $\sum a_n$ converge.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverge, then $\sum a_n$ diverge.

1c. Limit Comparison Test Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges

1d. Ratio Test Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

- If $\rho <$ 1 then $\sum a_n$ converges.
- If $\rho >$ 1 then $\sum a_n$ diverges.
- If $\rho =$ 1 then the test is inconclusive.

1e. Root Test Let $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If $\rho <$ 1 then $\sum a_n$ converges.
- If $\rho >$ 1 then $\sum a_n$ diverges.
- If $\rho =$ 1 then the test is inconclusive.

2. For problem 2, we now have an **alternating series**, i.e., $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

Alternating Series Test: If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

3. For problem 3, we now have an arbitrary series $\sum a_n$ (some terms might be positive, some might be negative, all might be positive, etc ...).

n^{th}**-term test** If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges .

4.

$$\lim_{n\to\infty} \frac{4n^3+6n^2-17n+9}{-5n^3+7n^2-9n-18} = -\frac{4}{5}$$

$$\frac{4n^3+6n^2-17n+9}{-5n^3+7n^2-9n-18} = \frac{4+\frac{6}{n}-\frac{17}{n^2}+\frac{9}{n^3}}{-5+\frac{7}{n}-\frac{9}{n^2}-\frac{18}{n^3}}$$

$$\xrightarrow{n\to\infty} \frac{+4+0-0+0}{-5+0-0-0} = -\frac{4}{5}$$

5.

$$\lim_{n\to\infty} \frac{7n^2+9}{-5n+2} = -\infty$$

Hint: watch your plus and minus.

$$\frac{7n^2+9}{-5n+2} = \frac{7+\frac{9}{n^2}}{-\frac{5}{n}+\frac{2}{n^2}} \xrightarrow{n\to\infty} \cancel{\frac{7+0}{0+0}} = \cancel{\frac{7}{0}} \overset{?}{=} \pm\infty$$

?

if n is really big, $\frac{7n^2+9}{-5n+2} = \frac{\text{a positive number}}{\text{a negative number}}$ = a neg. #

6.

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

Hint: L'Hopital

$$\lim_{x\to\infty} \frac{\ln x}{x} \underset{L'H}{\overset{\cancel{\infty/\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

7. Geometric Series

7a. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \boxed{\frac{1}{1-r}}$$

7b. Find the sum of the below series. (Note that the sum begins at $n = 10$ instead of $n = 0$.)

$$\sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-2} = \boxed{\frac{1}{3^7 \cdot 2}}$$

You only have to carry the algebra out as far as I indicated in class.

Notice if $|r|<1$, then

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots$$

(7b) $\displaystyle\sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-2}$ ← geometric series, $r = \frac{1}{3}$, $|r|<1$, converges

$$\sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-2} = \left(\frac{1}{3}\right)^8 + \left(\frac{1}{3}\right)^9 + \left(\frac{1}{3}\right)^{10} + \cdots$$

↑ want 1 here so factor out $\left(\frac{1}{3}\right)^8$

$$= \left(\frac{1}{3}\right)^8 \left[1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots \right]$$

$$= \left(\frac{1}{3}\right)^8 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$\overset{(7a)}{=} \frac{1}{3^8} \left[\frac{1}{1-\frac{1}{3}} \right] = \frac{1}{3^8} \left[\frac{1}{2/3} \right]$$

$$= \frac{1}{3^8} \cdot \frac{3}{2} = \frac{1}{3^7 \cdot 2}$$

On problems 8 - 10, check the correct box and then indicte your reasoning below. A correctly checked box without appropriate explanation will receive no points.

8. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{e}\right)^n$

Hint: $\frac{\pi}{e} \approx \frac{3.14}{2.7} \approx 1.16$.

☐ absolutely convergent

☐ conditionally convergent

☒ divergent

(a) abs. conv? no!

$$\sum_{n=1}^{\infty} \left| (-1)^n \left(\frac{\pi}{e}\right)^n \right| = \sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$$

diverges, geom. series, $r = \frac{\pi}{e}$, $|r| > 1$

(b) does $\sum (-1)^n \left(\frac{\pi}{e}\right)^n$ converge? No

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{e}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{\pi}{e}\right)^n$$

geom. series, $r = -\frac{\pi}{e}$, $|r| > 1$, divg.

9. $\sum_{n=17}^{\infty} (-1)^n \frac{1}{n!}$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

(a) abs conv ?

Consider $\sum_{n=17}^{\infty} \left| (-1)^n \frac{1}{n!} \right| = \sum_{n=17}^{\infty} \frac{1}{n!}$

Ratio Test "$a_n = n!$"

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1}$$

$$= \lim_{n\to\infty} \frac{n!}{(n!)(n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$\Downarrow$

Convergence

10. $\sum_{n=2}^{\infty} (-1)^n \frac{n^2}{n^3+8}$

- [ ] ~~absolutely convergent~~
- [x] conditionally convergent
- [ ] divergent

(a) abs. conv ? No

Consider $\sum_{n=2}^{\infty} \left| (-1)^n \frac{n^2}{n^3+8} \right| = \sum_{n=2}^{\infty} \frac{n^2}{n^3+8}$.

bs $a_n = \frac{n^2}{n^3+8} \overset{n \text{ big}}{\approx} \frac{n}{n^2} = \frac{1}{n} = b_n$

LCT. $L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^3+8} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3}{n^3+8} = 1$ $0 < \quad < \infty$

$\sum b_n = \sum \frac{1}{n} \implies$ diverge $\left\{ \begin{array}{l} \cdot \text{ p-series, } p=1,\ p \geq 1 \\ \text{or} \\ \cdot \text{ harmonic series} \end{array} \right.$

$\implies \sum_{n=2}^{\infty} \frac{n^2}{n^3+8}$ diverges

(b) Conditional conv ?

Consider $\sum_{n=2}^{\infty} (-1)^n \frac{n^2}{n^3+8}$ & use A.S.T.

- Is $a_n > a_{n+1}$? Yes

$\boxed{\begin{array}{c} 16 - x^3 < 0 \\ \Updownarrow \\ (16)^{1/3} < x \end{array}}$

$f(x) = \frac{x^2}{x^3+8} \implies$

$f'(x) = \frac{2x(x^3+8) - x^2(3x^2)}{(x^3+8)^2} = \frac{-x^4 + 16x}{(x^3+8)^2} = \frac{x(16-x^3)}{(x^3+8)^2} < 0$

$f'(x) < 0$ if x is big enough so $a_n > a_{n+1}$ if n is big enough

- $\lim_{n\to\infty} \frac{n^2}{n^3+8} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + 8/n^3} = \frac{0}{1+0} = 0$

so by AST, $\sum (-1)^n \frac{n^2}{n^3+8}$ converges