

MARK BOX	
PROBLEM	POINTS
1	10
2	10
3	10
4	10
5	10
6	10
7	10
8	10
9	10
10	10
%	100

NAME: KEY
 SSN: _____

Section 001 (MW 9:05)
 or
 Section 002 (MW 10:10)

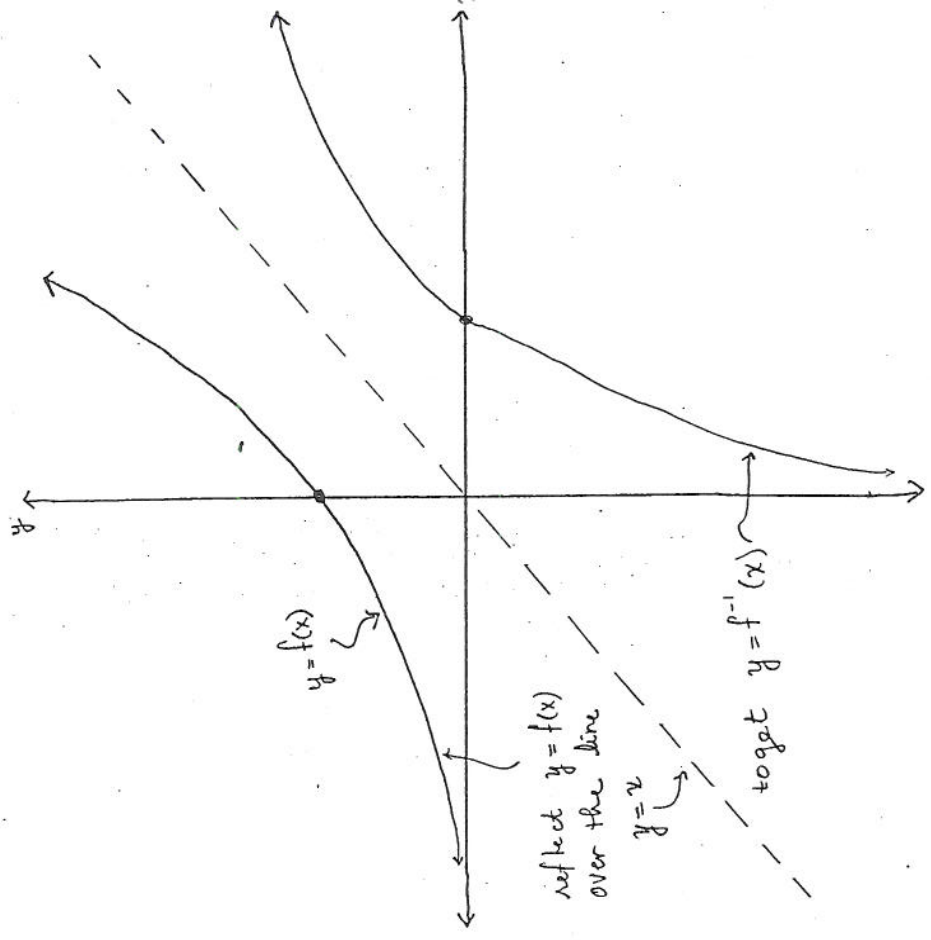
INSTRUCTIONS:

- To receive credit you must:
 - work in a logical fashion, show all your work, indicate your reasoning
 - when applicable put your answer on/in the line/box provided
 - if no such line/box is provided, then box your answer
- The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- You may **not** use a calculator, books, personal notes. Give exact answers: for example, write $\ln 2$ instead of .6931, write $\sqrt{2}$ instead of 1.414, write π instead of 3.1415, write $\frac{1}{3}$ instead of 0.3333.
- During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- This exam covers (from *Calculus* by Varberg, Purcell, Rigdon, 8th ed.): Chapter 7 (excluding 7.6), Section 8.1, and Section 8.2.

Problem Inspiration:

- an example from class
- homework problem § 7.4 # 5
- homework problem § 7.1 # 41
- homework problem § 7.9 # 13
- homework problem § 7.9 # 23
- homework problem § 8.1 # 17
- an example from class
- Prof. Kustin's Fall 2001 Math 142 Exam 1 # 4
- an example from class
- Take Home Quiz (warning: numbers are changed).

1. A function $y = f(x)$ is sketched below. Sketch the graph of its inverse function, i.e. sketch the graph of $y = f^{-1}(x)$, on the same grid.



2. Solve the equation

$$2 \log_9 \left(\frac{x}{3} \right) = 1$$

for x . Your answer should not have a logarithm nor exponential in it.

ANSWER: $x = 9$

WAY #1

$$2 \log_9 \left(\frac{x}{3} \right) = 1$$

$$\log_9 \left(\frac{x}{3} \right) = \frac{1}{2}$$

$$\frac{x}{3} = 9^{1/2}$$

$$\frac{x}{3} = 3$$

$$x = 9$$

WAY #2

$$2 \log_9 \left(\frac{x}{3} \right) = 1$$

$$\log_9 \left(\left(\frac{x}{3} \right)^2 \right) = 1$$

$$\left(\frac{x}{3} \right)^2 = 9^1$$

$$\frac{x}{3} = 3 \quad \text{--- not } x > 0.$$

$$x = 9$$

3a. For $x > 0$, the natural logarithm of x , i.e. $\ln x$, is defined by the definite integral

$$\ln x = \int_1^x \frac{dt}{t}$$

3b. Solve the equation

$$\int_1^x \frac{dt}{t} = 2 \int_1^x \frac{dt}{t}$$

for x . Your answer should not have a logarithm nor exponential in it.

ANSWER: $x = 3$

$$\int_{1/3}^x \frac{dt}{t} = 2 \int_1^x \frac{dt}{t}$$

$$\int_1^x \frac{dt}{t} + \int_1^{1/3} \frac{dt}{t} = 2 \int_1^x \frac{dt}{t}$$

$$-\int_1^{1/3} \frac{dt}{t} + \int_1^x \frac{dt}{t} = 2 \int_1^x \frac{dt}{t}$$

$$-\int_1^{1/3} \frac{dt}{t} = \int_1^x \frac{dt}{t} \quad \hookrightarrow$$

$$-\ln \left(\frac{1}{3} \right) = \ln x$$

$$\ln \left(\left(\frac{1}{3} \right)^{-1} \right) = \ln x$$

$$\ln 3 = \ln x$$

$$3 = x$$

4.

$$D_x [3 \ln(1 + e^{5x})] = \frac{15 e^{5x}}{1 + e^{5x}}$$

$$= 3 \cdot \frac{1}{1 + e^{5x}} \cdot D_x (1 + e^{5x})$$

$$= \frac{3}{1 + e^{5x}} \cdot (0 + e^{5x} D_x (5x))$$

$$= \frac{3}{1 + e^{5x}} \cdot (e^{5x} (5))$$

5.

$$D_x (x^{1+x}) = x^{1+x} \left(\ln x + \frac{1+x}{x} \right) \stackrel{\text{or}}{=} x^x (x \ln x + 1+x)$$

$$y = x^{1+x} \rightarrow \text{want } \frac{dy}{dx}$$

$$\ln y = \ln (x^{1+x})$$

$$\ln y = (1+x)(\ln x)$$

\rightarrow differentiate both sides with respect to x .

$$\frac{1}{y} \frac{dy}{dx} = [D_x (1+x)] \ln x + (1+x) D_x (\ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = 1 \cdot \ln x + \frac{(1+x)}{x}$$

$$\frac{dy}{dx} = y \left(\ln x + \frac{1+x}{x} \right)$$

$$= x^{1+x} \left(\ln x + \frac{1+x}{x} \right)$$

$$= x \cdot x^x \left(\frac{x \ln x + 1+x}{x} \right)$$

$$= x^x (x \ln x + 1+x)$$

5

6

$$\int \frac{3x^2 + 2x}{x+1} dx = \frac{3x^2}{2} - x + \ln|x+1| + C$$

$$x+1 \left| \begin{array}{r} 3x-1 \\ 3x^2+2x+0 \\ -(3x^2+3x) \end{array} \right.$$

$$\frac{-x+0}{-(-x-1)}$$

$$\int \frac{3x^2 + 2x}{x+1} dx = \int \left[3x-1 + \frac{1}{x+1} \right] dx$$

$$= \frac{3x^2}{2} - x + \int \frac{dx}{x+1}$$

$$\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}}$$

$$= \frac{3x^2}{2} - x + \int \frac{du}{u}$$

$$= \frac{3x^2}{2} - x + \ln|u| + C$$

$$= \frac{3x^2}{2} - x + \ln|x+1| + C$$

$$\int \sin^4(17x) \cos^3(17x) dx = \frac{1}{17} \left(\frac{\sin^5(17x)}{5} - \frac{\sin^7(17x)}{7} \right) + C$$

$$u = \cos(17x)$$

$$du = -17 \sin(17x) dx$$

$$v = \sin(17x)$$

$$dv = 17 \cos(17x) dx$$

or

$$\int \sin^4(17x) \cos^3(17x) dx =$$

$$= \frac{1}{17} \int \sin^4(17x) \cos^2(17x) [17 \cos(17x) dx]$$

$$= \frac{1}{17} \int \sin^4(17x) (1 - \sin^2(17x)) [17 \cos(17x) dx]$$

$$= \frac{1}{17} \int \sqrt{1-v^2} dv$$

$$= \frac{1}{17} \int (v^4 - v^6) dv$$

$$= \frac{1}{17} \left(\frac{v^5}{5} - \frac{v^7}{7} \right) + C$$

$$= \frac{1}{17} \left(\frac{\sin^5(17x)}{5} - \frac{\sin^7(17x)}{7} \right) + C$$

$$\underline{\underline{= \frac{\sin^5(17x)}{85} - \frac{\sin^7(17x)}{119} + C}}$$

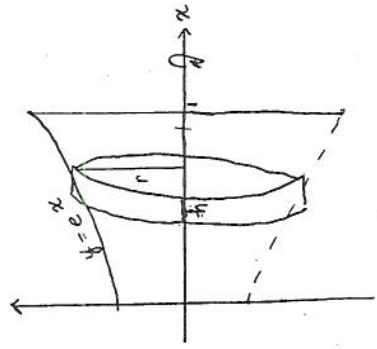
8. The volume of the solid generated by revolving the region bounded by:

$y = e^x$ and the x -axis and the y -axis and $x = 1$

$$\frac{\pi (e^2 - 1)}{2}$$

about the x -axis is

Volume of a typical element
 = (area of base) (height)
 = $(\pi r^2) (h)$
 = $\pi (e^x)^2 \Delta x$
 = $\pi e^{2x} \Delta x$



$$\text{Volume} = \int_0^1 \pi e^{2x} dx$$

$$= \pi \int_{x=0}^{x=1} e^{2x} dx$$

$$= \frac{\pi}{2} \int_{u=0}^{u=2} e^u du$$

$$= \frac{\pi}{2} (e^2 - e^0)$$

$$\begin{matrix} u = 2x & x=0 \Rightarrow u=0 \\ du = 2dx & x=1 \Rightarrow u=2 \end{matrix}$$

9. The solution to the differential equation

$$\frac{dy}{dt} = 12y(t)$$

$$y(t) = \frac{7e^{12t}}{e^{24}}$$

subject to the condition that $y(2) = 7$ is

$$\frac{dy}{dt} = 12y(t)$$

$$\frac{1}{y(t)} \frac{dy}{dt} = 12$$

$$D_t (\ln y(t)) = D_t (12t)$$

$$\ln y(t) = 12t + C$$

$$e^{\ln y(t)} = e^{12t + C}$$

$$y(t) = e^C \cdot e^{12t}$$

$$7 = y(2) = e^C \cdot e^{12 \cdot 2}$$

$$7 = e^C \cdot e^{24}$$

$$e^C = \frac{7}{e^{24}}$$

①

19. The rate of decay of a radioactive substance is proportional to the amount of such substance present.

Today we have P_0 grams of a radioactive substance.

Given that one-fourth of the substance decays every 7 years, how much will be left t years from today? Clearly explain your notation.

ANSWER: $P_0 e^{\frac{1}{7} \ln(\frac{3}{4}) t} = P_0 (\frac{3}{4})^{t/7}$ grams

Let t_0 = initial time = today

t = the # of years after t_0

$P(t)$ = the # of grams of the radioactive substance at time t .

$P_0 = P(0)$

① $\Rightarrow \frac{dP}{dt} = k P(t) \Rightarrow P(t) = P_0 e^{kt}$

Use ③ to find k :

③ $\frac{3}{4} P_0 = P(7) = P_0 e^{k(7)}$

$\Rightarrow P_0 e^{7k} = \frac{3}{4} P_0$

$\Rightarrow e^{7k} = \frac{3}{4}$

$\Rightarrow \ln e^{7k} = \ln(\frac{3}{4})$

$\Rightarrow 7k = \ln(\frac{3}{4})$

$\Rightarrow k = \frac{1}{7} \ln(\frac{3}{4})$

$P(t) = P_0 e^{\frac{1}{7} \ln(\frac{3}{4}) t}$
 $= P_0 [e^{\frac{1}{7} \ln(\frac{3}{4})}]^t$
 $= P_0 [e^{\ln(\frac{3}{4})^{1/7}}]^t$
 $= P_0 [(\frac{3}{4})^{1/7}]^t$
 $= P_0 (\frac{3}{4})^{t/7}$