

1.1. Vector - determined by its magnitude (i.e. length) and direction.

Remark: a POINT $P = (a, b, c)$ determines a VECTOR $\vec{OP} = \langle a, b, c \rangle$

1.2. Products Given: $\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$

$$0 \leq \angle \vec{A}\vec{B} \stackrel{\text{def}}{=} \text{angle between } \vec{A} \text{ \& } \vec{B} \stackrel{\text{def}}{=} \theta_{AB} \leq \pi$$

$\vec{n}_{AB} \stackrel{\text{def}}{=} \text{the right-hand-rule unit vector } \perp \text{ to } \vec{A} \text{ and } \vec{B}$

direction angles $0 \leq \alpha, \beta, \gamma \leq \pi$ for \vec{A} .

$$\begin{aligned} \text{Then:} \\ \text{length of } \vec{A} &= \|\vec{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{A} \cdot \vec{A}} \\ \text{dot product} &= \vec{A} \cdot \vec{B} = \|\vec{A}\| \|\vec{B}\| \cos \theta_{AB} = a_1 b_1 + a_2 b_2 + a_3 b_3 \\ \text{cross product} &= \vec{A} \times \vec{B} = \left[\|\vec{A}\| \|\vec{B}\| \sin \theta_{AB} \right] \vec{n}_{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ \text{triple product} &= \left[\vec{A}, \vec{B}, \vec{C} \right] = \vec{A} \cdot \vec{B} \cdot \vec{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

$$\text{angle btw.} = \angle \vec{A}\vec{B} = \cos^{-1} \left(\frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\| \|\vec{B}\|} \right)$$

$$\alpha = \angle \vec{A}\vec{i} = \cos^{-1} \left(\frac{a_1}{\|\vec{A}\|} \right)$$

$$\beta = \angle \vec{A}\vec{j} = \cos^{-1} \left(\frac{a_2}{\|\vec{A}\|} \right)$$

$$\gamma = \angle \vec{A}\vec{k} = \cos^{-1} \left(\frac{a_3}{\|\vec{A}\|} \right)$$

$$\text{so } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

1.3. 1,2 in action $\vec{A} \neq \vec{0}$ and $\vec{B} \neq \vec{0}$

\vec{A} is a UNIT vector $\Leftrightarrow \|\vec{A}\| = 1$

the NORMALIZED vector of \vec{A} is $\frac{\vec{A}}{\|\vec{A}\|}$

\vec{A} & \vec{B} are ORTHONORMAL $\Leftrightarrow \vec{A} \perp \vec{B}$ and $\|\vec{A}\| = 1 = \|\vec{B}\|$

$\vec{A} \perp \vec{B} \Leftrightarrow \vec{A} \cdot \vec{B}$ are PERPENDICULAR $\Leftrightarrow \vec{A} \cdot \vec{B} = 0 \Leftrightarrow \vec{A}$ & \vec{B} are ORTHOGONAL

$\|\vec{A}\| \|\vec{B}\| \Leftrightarrow \vec{A} \times \vec{B}$ are PARELLEL $\Leftrightarrow \vec{A} \times \vec{B} = \vec{0} \Leftrightarrow \vec{A} = k\vec{B}$ for some $k \in \mathbb{R}$

the area of the triangle determined by \vec{A} and \vec{B} is $\text{Area} = \frac{1}{2} \|\vec{A} \times \vec{B}\|$

the area of the parallelogram determined by \vec{A} and \vec{B} is $\text{Area} = \|\vec{A} \times \vec{B}\|$

the volume of the parallelepiped determined by $\vec{A}, \vec{B}, \vec{C}$ is $\text{Vol} = \left| \left[\vec{A}, \vec{B}, \vec{C} \right] \right|$

1.4. Algebraic Properties

$$\begin{aligned} (s\vec{A}) \cdot \vec{B} &= s(\vec{A} \cdot \vec{B}) = \vec{A} \cdot (s\vec{B}) & \Leftrightarrow & (s\vec{A}) \times \vec{B} = s(\vec{A} \times \vec{B}) = \vec{A} \times (s\vec{B}) \\ \vec{A} \cdot \vec{B} &= \vec{B} \cdot \vec{A} & \Leftrightarrow & \vec{A} \times \vec{B} = -(\vec{B} \times \vec{A}) \end{aligned}$$

$$(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C} \quad \Leftrightarrow \quad (\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$$

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \quad \Leftrightarrow \quad \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

1.5. Components

The parallel-perpendicular decomposition of \vec{B} is $\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$ where:

$$\vec{B}_{\parallel} = \frac{\vec{A} \cdot \vec{B}}{\vec{A} \cdot \vec{A}} \vec{A} = \left(\frac{\vec{B} \cdot \vec{A}}{\|\vec{A}\|} \right) \frac{\vec{A}}{\|\vec{A}\|} \quad \text{is } \parallel \text{ to } \vec{A}$$

$$\vec{B}_{\perp} = \frac{(\vec{A} \times \vec{B}) \times \vec{A}}{\vec{A} \cdot \vec{A}} = \vec{B} - \vec{B}_{\parallel} \quad \text{is } \perp \text{ to } \vec{A}$$

\vec{B}_{\parallel} is called the orthogonal projection of \vec{B} onto \vec{A} .

The (signed) component of \vec{B} along \vec{A} is:

$$\text{comp}_{\vec{A}} \vec{B} \stackrel{\text{def}}{=} \begin{cases} + \|\vec{B}_{\parallel}\| & \text{if } 0 \leq \theta_{AB} \leq \pi/2 \\ - \|\vec{B}_{\parallel}\| & \text{if } \pi/2 \leq \theta_{AB} \leq \pi \end{cases} \stackrel{\text{L,2}}{=} \|\vec{B}\| \cos \theta_{AB} \stackrel{\text{L,2}}{=} \vec{B} \cdot \frac{\vec{A}}{\|\vec{A}\|}$$

1.6. Equations and parameterizations

\triangleright The line \mathcal{L} through $R_0 = (x_0, y_0, z_0)$ and parallel to $\vec{v}_0 = \langle a, b, c \rangle$:

$$\vec{R}(t) = \vec{R}_0 + t\vec{v}_0 \Leftrightarrow \begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases} \Leftrightarrow \begin{cases} \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \\ \text{if } abc \neq 0 \end{cases}$$

\triangleright The plane \mathcal{P} through $R_0 = (x_0, y_0, z_0)$ and with normal $\vec{n} = \langle a, b, c \rangle$:

$$\begin{aligned} \langle x, y, z \rangle - \vec{R}_0 \cdot \vec{n} &= 0 \quad \Leftrightarrow \quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \\ &\Leftrightarrow \quad ax + by + cz = d \quad \text{where } d = ax_0 + by_0 + cz_0 \end{aligned}$$

\triangleright For a plane \mathcal{P} given by $ax + by + cz = d$:

$$d(\mathcal{P}, (x_1, y_1, z_1)) = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

\triangleright For radius ρ , center R_0 , and a right-handed system of orthonormal vector $\vec{e}_1, \vec{e}_2, \vec{e}_3$:

$$\vec{R}(t) = \vec{R}_0 + \cos t \rho \vec{e}_1 + \sin t \rho \vec{e}_2 + at \vec{e}_3$$

o if $a = 0$: is a circle in the plane determined by \vec{e}_1 and \vec{e}_2

o if $a > 0$: is a right-handed helix with pitch $2\pi|a|$ and axis parallel to \vec{e}_3

o if $a < 0$: is a left-handed helix with pitch $2\pi|a|$ and axis parallel to \vec{e}_3

1.7. Force: For a constant force \vec{F} acting thru a displacement \vec{D}

▷ The work W done by \vec{F} is $W = \vec{D} \cdot \vec{F}$;

▷ The torque τ due to \vec{F} at the point D is $\tau = \vec{D} \times \vec{F}$.

1.8. Cylindrical Coordinates (r, θ, z) of (x, y, z) :

$$r \geq 0 \quad 0 \leq \theta < 2\pi \quad -\infty < z < \infty$$

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$r = \sqrt{x^2 + y^2} \quad \theta = \begin{cases} \tan^{-1}(y/x) & \text{if } (x, y) \in 1^{\text{st}} \text{ quad} \\ \pi + \tan^{-1}(y/x) & \text{if } (x, y) \in 2/3^{\text{rd}} \text{ quad} \\ 2\pi + \tan^{-1}(y/x) & \text{if } (x, y) \in 4^{\text{th}} \text{ quad} \end{cases}$$

1.9. Spherical Coordinates (ρ, θ, ϕ) of (x, y, z) : note $r = \rho \sin \phi$

$$\begin{aligned} \rho &\geq 0 & 0 &\leq \theta < 2\pi & 0 &\leq \phi \leq \pi \\ x &= \rho \sin \phi \cos \theta & y &= \rho \sin \phi \sin \theta & z &= \rho \cos \phi \\ \rho &= \sqrt{x^2 + y^2 + z^2} & \theta &= \text{as above} & \phi &= \cos^{-1} \frac{z}{\rho} \end{aligned}$$

Figure 1.4.2

Representing a point (x, y, z) in Cartesian coordinates in its cylindrical coordinates (r, θ, z) .

Figure 1.4.5

Representing a point (x, y, z) in Cartesian coordinates in its spherical coordinates (ρ, θ, ϕ) .

2.1. Defs.

- a. the neighborhood of $p \in \mathbb{R}^n$ of radius r is $N_r(p) = \{q \in \mathbb{R}^n : \|p - q\| < r\}$
- b. open set - pg 94

2.2. Set-up

Given: an open subset $U^n \subset \mathbb{R}^n$

a function $\vec{f}: U^n \rightarrow \mathbb{R}^m$.

So elts. of U^n look like $\vec{x} = \langle x_1, \dots, x_n \rangle$ and $\vec{x}_0 = \langle x_{01}, \dots, x_{0n} \rangle$

and \vec{f} has the form

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}_{m \times 1}$$

where $f_i: U^n \rightarrow \mathbb{R}^1$.

► **Limits (pg 98) and Continuity (pg 104)**

$\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b} \iff \vec{f}(\vec{x}) \approx \vec{b}$ provided $\vec{x} \approx \vec{x}_0$

$\iff \lim_{\vec{x} \rightarrow \vec{x}_0} \langle f_1(\vec{x}), \dots, f_m(\vec{x}) \rangle = \langle b_1, \dots, b_m \rangle$

$\iff \lim_{\vec{x} \rightarrow \vec{x}_0} f_i(\vec{x}) = b_i$ for each i .

\vec{f} is continuous at $\vec{x}_0 \iff \lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{f}(\vec{x}_0)$

$\iff \lim_{\vec{x} \rightarrow \vec{x}_0} \langle f_1(\vec{x}), \dots, f_m(\vec{x}) \rangle = \langle f_1(\vec{x}_0), \dots, f_m(\vec{x}_0) \rangle$

$\iff \lim_{\vec{x} \rightarrow \vec{x}_0} f_i(\vec{x}) = f_i(\vec{x}_0)$ for each i

\iff each f_i is continuous at \vec{x}_0 .

► **Partial Derivatives of the f_i 's and the class C^k**

Def: An *iterated partial derivative* (i.p.d.) of f_i of order k is of the form:

$$\frac{\partial^k f_i}{\partial x_{i_1} \dots \partial x_{i_k}} = \frac{\partial}{\partial x_{i_k}} \left(\dots \left(\frac{\partial}{\partial x_{i_2}} \left(\frac{\partial f_i}{\partial x_{i_1}} \right) \right) \right)$$

Def: \vec{f} is (in the class) C^k at $\vec{x}_0 \in U^n$

\iff each of the k^{th} -ordered i.p.d. of each f_i exists and is continuous in a neighborhood of \vec{x}_0 .

Def: \vec{f} is (in the class) C^k on U^n

$\iff \vec{f}$ is C^k at each $\vec{x} \in U^n$

\iff each of the k^{th} -ordered i.p.d. of each f_i exists and is continuous at each $\vec{x} \in U^n$.

Euler's Theorem 3.1: If $f_i \in C^2$ at $\vec{x}_0 \in U^n$, then $\frac{\partial^2 f_i(\vec{x}_0)}{\partial x_{i_2} \partial x_{i_1}} = \frac{\partial^2 f_i(\vec{x}_0)}{\partial x_{i_1} \partial x_{i_2}}$

► Derivative of f

Form: the matrix of partial derivatives of \vec{f} ,

which is also called the derivative of \vec{f} (at \vec{x}_0):

$$D_M \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n} \rightsquigarrow D_M \vec{f}(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_1(\vec{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_m(\vec{x}_0)}{\partial x_n} \end{bmatrix}_{m \times n}$$

and consider

$$D_M \vec{f}(\vec{x}_0) \otimes (\vec{x} - \vec{x}_0) \stackrel{\text{def}}{=} \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_1(\vec{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_m(\vec{x}_0)}{\partial x_n} \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 - x_{01} \\ \vdots \\ x_n - x_{0n} \end{bmatrix}_{n \times 1} \\ = \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1} (x_1 - x_{01}) + \cdots + \frac{\partial f_1(\vec{x}_0)}{\partial x_n} (x_n - x_{0n}) \\ \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1} (x_1 - x_{01}) + \cdots + \frac{\partial f_m(\vec{x}_0)}{\partial x_n} (x_n - x_{0n}) \end{bmatrix}_{m \times 1}$$

Then \vec{f} is differentiable at \vec{x}_0 if the partials $\frac{\partial f_i}{\partial x_j}$ of \vec{f} exist at \vec{x}_0 and if

$$\lim_{\vec{x} \rightarrow \vec{x}_0} \frac{\|\vec{f}(\vec{x}) - \left[\vec{f}(\vec{x}_0) + D_M \vec{f}(\vec{x}_0) \otimes (\vec{x} - \vec{x}_0) \right]\|}{\|\vec{x} - \vec{x}_0\|} = 0. \quad (2.2')$$

When this is the case, if $\vec{x} \approx \vec{x}_0$, then:

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_0) + D_M \vec{f}(\vec{x}_0) \otimes (\vec{x} - \vec{x}_0), \quad (2.2'')$$

i.e.,

$$\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_0) \\ \vdots \\ f_m(\vec{x}_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1} (x_1 - x_{01}) + \cdots + \frac{\partial f_1(\vec{x}_0)}{\partial x_n} (x_n - x_{0n}) \\ \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1} (x_1 - x_{01}) + \cdots + \frac{\partial f_m(\vec{x}_0)}{\partial x_n} (x_n - x_{0n}) \end{bmatrix}$$

••• § 2.3 Theorem 9: If $\vec{f} \in C^1$ at $\vec{x}_0 \in U^n$, then \vec{f} is differentiable at \vec{x}_0 .

If $\vec{f} \in C^1$ at each $\vec{x}_0 \in U^n$, then \vec{f} is differentiable on U^n .

2.3. Special Cases of (2.2') $f: \mathbb{R} \rightarrow \mathbb{R}$ so: $y = f(x)$ and $D_M f = \left[\frac{df}{dx} \right]$.

If $x \approx x_0$ then:

$$f(x) \approx f(x_0) + \left[\frac{df(\vec{x}_0)}{dx} \right] \otimes [x - x_0] \\ \approx f(x_0) + \frac{df(x_0)}{dx} (x - x_0).$$

Special Cases of (2.2''): $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ so: $z = f(x, y)$ and $D_M f = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right]$.

If $(x, y) \approx (x_0, y_0)$ then:

$$f(x, y) \approx f(x_0, y_0) + \left[\frac{\partial f(x_0, y_0)}{\partial x} \quad \frac{\partial f(x_0, y_0)}{\partial y} \right] \otimes \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\ \approx f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x} (x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y} (y - y_0).$$

2.4. Differentiation Rules

Given: an open subset $U^n \subset \mathbb{R}^n$ and $\vec{x}_0 \in U^n$

functions $\vec{f}: U^n \rightarrow \mathbb{R}^m$ and $\vec{g}: U^n \rightarrow \mathbb{R}^p$ that are each differentiable at \vec{x}_0
a constant $c \in \mathbb{R}$

Then:

$$D_M [c \vec{f}] (\vec{x}_0) = c D_M [\vec{f}(\vec{x}_0)] \\ D_M [\vec{f} + \vec{g}] (\vec{x}_0) = D_M [\vec{f}(\vec{x}_0)] + D_M [\vec{g}(\vec{x}_0)] \\ D_M [\vec{f} \vec{g}] (\vec{x}_0) \stackrel{\text{m=1}}{=} D_M [\vec{f}(\vec{x}_0)] \vec{g}(\vec{x}_0) + \vec{f}(\vec{x}_0) D_M [\vec{g}(\vec{x}_0)] \\ D_M \left[\frac{\vec{f}}{\vec{g}} \right] (\vec{x}_0) \stackrel{\text{m=1}}{\vec{g}(\vec{x}_0) \neq 0}{=} \frac{D_M [\vec{f}(\vec{x}_0)] \vec{g}(\vec{x}_0) - \vec{f}(\vec{x}_0) D_M [\vec{g}(\vec{x}_0)]}{[\vec{g}(\vec{x}_0)]^2}.$$

Given: open subsets $U^n \subset \mathbb{R}^n$ and $V^m \subset \mathbb{R}^m$ and $\vec{x}_0 \in U^n$

functions $\vec{g}: U^n \rightarrow \mathbb{R}^m$ and $\vec{f}: V^m \rightarrow \mathbb{R}^p$

$\vec{g}(U^n) \subset V^m$ and so it makes sense to have $\vec{f} \circ \vec{g}: U^n \rightarrow \mathbb{R}^p$.

\vec{g} is differentiable at \vec{x}_0 and \vec{f} is differentiable at $\vec{g}(\vec{x}_0)$.

Then the Chain Rule says that :

$$D_M (\vec{f} \circ \vec{g}) (\vec{x}_0) = D_M \vec{f} (\vec{g}(\vec{x}_0)) \otimes D_M \vec{g}(\vec{x}_0).$$

2.5. Vector-Valued Functions

Given: an subset $\mathbb{D}^1 \subset \mathbb{R}^1$ and $t_0 \in \mathbb{D}^1$
 a function $\vec{F}: \mathbb{D}^1 \rightarrow \mathbb{R}^m$

so \vec{F} has the form $\vec{F}(t) = \langle f_1(t), \dots, f_m(t) \rangle$ where $f_i: \mathbb{D}^1 \rightarrow \mathbb{R}$.

\vec{F} is called a VECTOR-VALUED FUNCTION.

One thinks of \vec{F} as tracing out (ie. parameterizing) a (1-D) space curve \mathcal{C} in \mathbb{R}^m .

When \mathbb{D}^1 is an open set:

$$D_M \vec{F} = \begin{bmatrix} \frac{df_1}{dt} \\ \vdots \\ \frac{df_m}{dt} \end{bmatrix}$$

and (2.2') becomes:

$$\vec{F} \text{ is differentiable at } t_0 \iff \lim_{t \rightarrow t_0} \frac{\vec{F}(t) - \vec{F}(t_0)}{t - t_0} \text{ exists}$$

and in this case,

$$\frac{d\vec{F}}{dt} \Big|_{t_0} = \vec{F}'(t_0) = \lim_{t \rightarrow t_0} \frac{\vec{F}(t) - \vec{F}(t_0)}{t - t_0} = \langle f_1'(t_0), \dots, f_m'(t_0) \rangle.$$

2.6. Paths/Curves

- ▷ A PATH is a vector-valued function $\vec{r}: [a, b] \rightarrow \mathbb{R}^n$. Usually, $m = 2$ or 3 .
- ▷ The CURVE or ARC \mathcal{C} that is PARAMETERIZED by \vec{r} is $\mathcal{C} = \{\vec{r}(t) : a \leq t \leq b\}$.
- ▷ The unit tangent vector to \mathcal{C} at t_0 is $\vec{T} = \frac{\vec{r}'(t_0)}{\|\vec{r}'(t_0)\|}$.
- ▷ If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the arc length of \mathcal{C} is: $s(a, b) = \int_a^b ds = \int_a^b \|\vec{r}'(t)\| dt$
 $w/ ds = \vec{r}'(t) dt = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle dt$ & $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

- ▷ If: $I \subset [a, b]$
 then: \vec{r} is 1-to-1 on $I \iff$ the curve $\{\vec{r}(t) : t \in I\}$ does not intersect itself.

\mathcal{C} is a SIMPLE CURVE if it has a parameterization \vec{r} where:

$$(1) \vec{r} \text{ is piecewise } C^1 \text{ on } [a, b] \quad (2) \vec{r} \text{ is 1-to-1 on } [a, b].$$

\mathcal{C} is a CLOSED CURVE if $\vec{r}(a) = \vec{r}(b)$.

\mathcal{C} is a SIMPLE CLOSED CURVE if it has a parameterization \vec{r} where:

$$(1) \vec{r} \text{ is piecewise } C^1 \text{ on } [a, b] \quad (2) \vec{r} \text{ is 1-to-1 on } [a, b] \\ (3) \vec{r}(a) = \vec{r}(b).$$

\mathcal{C} is a SMOOTH if it has a parameterization \vec{r} where:

$$(1) \vec{r} \text{ is } C^1 \text{ on } [a, b] \quad (2) \vec{r} \text{ is 1-to-1 on } [a, b] \\ (3) \vec{r}' \neq \vec{0} \text{ on } [a, b].$$

2.7. Differentiation Rules for vector-valued functions

Given: an open subset $\mathbb{D}^1 \subset \mathbb{R}^1$

differentiable functions $\vec{F}, \vec{G}: \mathbb{D}^1 \rightarrow \mathbb{R}^n$ and $p: \mathbb{D}^1 \rightarrow \mathbb{R}$

Then: $\frac{d}{dt} [p(t) \vec{F}(t)] = p'(t) \vec{F}(t) + p(t) \vec{F}'(t)$

$\frac{d}{dt} [\vec{F}(t) \cdot \vec{G}(t)] = \vec{F}'(t) \cdot \vec{G}(t) + \vec{F}(t) \cdot \vec{G}'(t)$

$\frac{d}{dt} [\vec{F}(t) \times \vec{G}(t)] = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)$.

Chain Rule: $\frac{d}{dt} [\vec{F}(p(t))] = p'(t) \vec{F}'(p(t))$.

2.8. Motion For a particle moving through space on a "nice" curve \mathcal{C} with :

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ position vector

$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ velocity vector

$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}$ acceleration vector

$v(t) = \|\vec{v}(t)\| \neq \frac{d\|\vec{r}(t)\|}{dt}$ speed function

$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$ unit tangent vector to \mathcal{C} at the pt. $\vec{r}(t)$

$\vec{N}(t) = \frac{d\vec{T}(t)/dt}{\|d\vec{T}(t)/dt\|}$ unit principle normal vector to \mathcal{C} at the pt. $\vec{r}(t)$

$\vec{I}(t) \perp \vec{N}(t)$ with $\vec{I}(t) \parallel \vec{v}(t)$ & $\vec{N}(t) \perp \vec{v}(t)$

$\vec{a}(t) = a_T(t) \vec{T}(t) + a_N(t) \vec{N}(t)$ with $a_T(t) = \frac{dv(t)}{dt}$ & $a_N(t) = v(t) \left\| \frac{d\vec{T}(t)}{dt} \right\|$

$\|\vec{a}(t)\|^2 = |a_T(t)|^2 + |a_N(t)|^2$ & $a_N \geq 0$

$\vec{F} = m\vec{a}$ if a force \vec{F} is acting on the particle of mass m

2.9. Scalar Fields

Given: a subset $\mathbb{D}^n \subset \mathbb{R}^n$ and a function $f: \mathbb{D}^n \rightarrow \mathbb{R}$

so f has the form $f(\vec{x}) = w$, or equiv. $f(x_1, \dots, x_n) = w$

Then: f is called a SCALAR FIELD

the GRAPH of $f = \{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, \dots, x_n) \in \mathbb{D}^n\}$

the LEVEL SET of value $c \in \mathbb{R}$ for f is $\{\vec{x} \in \mathbb{D}^n : f(\vec{x}) = c\} \subset \mathbb{R}^n$

▷ $[n = 2] \Rightarrow$ [level set = LEVEL CURVE]

▷ $[n = 3] \Rightarrow$ [level set = LEVEL SURFACE]

$$D_M f = \left[\frac{\partial f}{\partial x_1} \quad \dots \quad \frac{\partial f}{\partial x_n} \right]$$

the GRADIENT of $f = \vec{\nabla} f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$.

2.10. Curves on Surfaces

- ★ Given: an open set $\mathbb{D}^n \subset \mathbb{R}^n$ and a scalar field $f: \mathbb{D}^n \rightarrow \mathbb{R}$
 $f(x_1, \dots, x_n) = w$
 an interval $I \subset \mathbb{R}$ and a path $\vec{r}: I \rightarrow \mathbb{R}^n$
 $\vec{r}(t) = (r_1(t), \dots, r_n(t))$
 $F \stackrel{\text{def}}{=} f \circ \vec{r}: I \rightarrow \mathbb{R}$
 $\vec{r}(I) \subset \mathbb{D}^n$ so $F(t) = f(r_1(t), \dots, r_n(t))$

When $n = 2$, think of F as a curve on the surface of f .

If f and \vec{r} are nice, ie. differentiable, then

$$D_M f = \left[\frac{\partial f}{\partial x_1} \quad \dots \quad \frac{\partial f}{\partial x_n} \right] \quad \text{and} \quad D_M \vec{r} = \begin{bmatrix} \frac{dr_1}{dt} \\ \vdots \\ \frac{dr_n}{dt} \end{bmatrix}$$

and the Chain Rule (2.4) gives:

$$\frac{dF}{dt} \Big|_{t_0} = D_M f(\vec{r}(t_0)) \otimes D_M \vec{r}(t_0) = \vec{\nabla} f \Big|_{\vec{r}(t_0)} \cdot \vec{r}'(t_0).$$

- ▶ In ★, consider the level set S given by $f(\vec{x}) = k$:
 If: $\vec{r}(I) \subset S$ so $(f \circ \vec{r})(t) = k$ for each $t \in I$
 $\vec{x}_0 \in S$ so $f(\vec{x}_0) = k$
 $t_0 \in I$ with $\vec{r}(t_0) = \vec{x}_0$
 Then: $\vec{\nabla} f \Big|_{\vec{x}_0} \cdot \vec{r}'(t_0) = \frac{d}{dt} (f \circ \vec{r}) \Big|_{t_0} = 0$
 so $\vec{\nabla} f \Big|_{\vec{x}_0} \perp$ (the tangent vector to $\vec{r}(t)$) $\Big|_{t_0}$
 so $\vec{\nabla} f \Big|_{\vec{x}_0} \perp S \Big|_{\vec{x}_0}$.
- ▶ In ★, consider the case where \vec{r} is a line segment through \vec{x}_0 in the direction of \vec{u} :
 Given: $\vec{x}_0 \in \mathbb{D}^n$ and $\vec{u} \in \mathbb{R}^n$ and so $\vec{r}(t) = \vec{x}_0 + t\vec{u}$.

The DIRECTIONAL DERIVATIVE of f at \vec{x}_0 along \vec{u} is :

$$\begin{aligned} D_{\vec{u}} f(\vec{x}_0) &\stackrel{\text{def}}{=} \frac{d(f \circ \vec{r})}{dt} \Big|_{\substack{t=0 \\ \vec{r}=\vec{x}_0}} = \lim_{h \rightarrow 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h} \\ &= \vec{u} \cdot \vec{\nabla} f \Big|_{\vec{x}_0} = \|\vec{u}\| \left\| \vec{\nabla} f \Big|_{\vec{x}_0} \right\| \cos \angle \vec{u} \vec{\nabla} f \Big|_{\vec{x}_0} \end{aligned}$$

- When $\|\vec{u}\| = 1$, since $D_{\vec{u}} f \Big|_{\vec{x}_0}$ is just the rate of change of f at \vec{x}_0 in the direction of \vec{u} , $D_{\vec{u}} f \Big|_{\vec{x}_0}$ is called the directional derivative of f at \vec{x}_0 in the direction of \vec{u} .
 $\pm \vec{\nabla} f$ points in the direction of the maximum rate of $\left[\begin{smallmatrix} \text{increase} \\ \text{decrease} \end{smallmatrix} \right]$ of f .
 $\pm \left\| \vec{\nabla} f \right\|$ equals the maximum rate of $\left[\begin{smallmatrix} \text{increase} \\ \text{decrease} \end{smallmatrix} \right]$ of f per unit distance.

2.11. Tangent Planes

- Here: open sets $\mathbb{D}^n \subset \mathbb{R}^n$
 ▶ Given: a scalar field $f: \mathbb{D}^3 \rightarrow \mathbb{R}$, so f has the form $w = f(x, y, z)$
 $\vec{x}_0 = \langle x_0, y_0, z_0 \rangle \in \mathbb{D}^3$ with $f(\vec{x}_0) = k$
 the level surface S given by $f(\vec{x}) = k$, so $\vec{x}_0 \in S$
 $\vec{\nabla} f \Big|_{\vec{x}_0} \neq 0$.
 Then: $\vec{\nabla} f \Big|_{\vec{x}_0} \perp S \Big|_{\vec{x}_0}$
 the eq. of the tangent plane to S at the point (x_0, y_0, z_0) is :
 $\vec{\nabla} f \Big|_{\vec{x}_0} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.
 ▶ Given: a scalar field $g: \mathbb{D}^2 \rightarrow \mathbb{R}$, so g has the form $z = g(x, y)$
 Let: $f(x, y, z) = g(x, y) - z$
 So: [the level surface $f(x, y, z) = 0$] = [the graph (ie. surface) of $z = g(x, y)$] $\stackrel{\text{def}}{=} S$
 So: $\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle \Big|_{(x_0, y_0)} \perp S \Big|_{(x_0, y_0, g(x_0, y_0))}$
 the eq. of the tangent plane to S at the point $(x_0, y_0, g(x_0, y_0))$ is both :
 $\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle \Big|_{(x_0, y_0)} \cdot \langle x - x_0, y - y_0, z - g(x_0, y_0) \rangle = 0$
 $z = g(x_0, y_0) + \frac{\partial g}{\partial x} \Big|_{(x_0, y_0)} (x - x_0) + \frac{\partial g}{\partial y} \Big|_{(x_0, y_0)} (y - y_0)$.
- ▶ Given: a scalar field $h: \mathbb{D}^1 \rightarrow \mathbb{R}$, so h has the form $y = h(x)$
 Let: $f(x, y, z) = h(x) - y$
 So: [the level surface $f(x, y, z) = 0$ intersected with the plane $z = 0$]
 $=$ [the graph of $y = h(x)$] $\stackrel{\text{def}}{=} G$
 So: $\langle \frac{dh}{dx}, -1, 0 \rangle \Big|_{(x_0)} \perp \mathcal{G} \Big|_{(x_0, h(x_0), 0)}$
 the equation of the tangent line to \mathcal{G} at the point $(x_0, h(x_0), 0)$ is both :
 $\langle \frac{dh}{dx}, -1, 0 \rangle \Big|_{(x_0)} \cdot \langle x - x_0, y - h(x_0), 0 \rangle = 0$
 $y = h(x_0) + \frac{dh}{dx} \Big|_{(x_0)} (x - x_0)$ and $z = 0$.

§4.3-4.4 Vector Fields and Div and Curl

4.1. Vector Fields

Given: a subset $\mathbb{D}^n \subset \mathbb{R}^n$

a function $\vec{F}: \mathbb{D}^n \rightarrow \mathbb{R}^n$

so \vec{F} has the form $\vec{F}(x_1, \dots, x_n) = \langle F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n) \rangle$

where $F_i: \mathbb{D}^n \rightarrow \mathbb{R}$.

Then: \vec{F} is called a vector field

visualize \vec{F} as attaching the vector $\vec{F}(\vec{x})$ to the point \vec{x}

F_i are called the component scalar fields

Given: a path $\vec{c}: \mathbb{D}^1 \rightarrow \mathbb{R}^n$

so \vec{c} has the form $\vec{c}(t) = \langle x_1(t), \dots, x_n(t) \rangle$ where $x_i: \mathbb{D}^1 \rightarrow \mathbb{R}$.

If: $\vec{c}'(t) = \vec{F}(\vec{c}(t))$

Then: \vec{c} is called a flow line of \vec{F}

\vec{c} is the sol'n to the system of differential eqs: $x_i'(t) = F_i(x_1(t), \dots, x_n(t))$

$\vec{F}(\vec{c}(t_0))$ is tangent to the curve \vec{c} at the point $\vec{c}(t_0)$

$n = 3$: if \vec{F} is the velocity field of a fluid, then \vec{c} is the path followed by a particle in the fluid.

4.2. Divergence & Curl

Def's: the vector differential operator

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

the Laplacian operator

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec{\nabla} \cdot \vec{\nabla}$$

Let: $\begin{bmatrix} S \\ \mathcal{V} \end{bmatrix}$ be the class of all the (nice) $\begin{bmatrix} \text{scalar} \\ \text{vector} \end{bmatrix}$ fields from $\mathbb{D}^3 \subset \mathbb{R}^3$ to $\begin{bmatrix} \mathbb{R} \\ \mathbb{R}^3 \end{bmatrix}$.

Def's: for $f \in S$ and $\vec{F} \in \mathcal{V}$

$$\overrightarrow{\text{grad}} f = \vec{\nabla}(f) \quad ; \quad \text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} \quad ; \quad \overrightarrow{\text{curl}} \vec{F} = \vec{\nabla} \times \vec{F}.$$

So:

$$\begin{array}{ccc} S & \xrightarrow{\overrightarrow{\text{grad}}} & \mathcal{V} & \xrightarrow{\overrightarrow{\text{curl}}} & S \\ S & \xrightarrow{\nabla^2} & S & \text{ and } & \mathcal{V} & \xrightarrow{\nabla^2} & \mathcal{V} \end{array}$$

Given: a function $\vec{F}: \mathbb{D}^2 \rightarrow \mathbb{R}^2$

so \vec{F} has the form $\vec{F} = \langle F_1, F_2 \rangle$

Then: scalar curl of \vec{F} = $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

4.3. Basic Identities

1. $\vec{\nabla}(f + g) = \vec{\nabla}f + \vec{\nabla}g$
2. $\vec{\nabla}(cf) = c \vec{\nabla}f$ where $c \in \mathbb{R}$
3. $\vec{\nabla}(fg) = f \vec{\nabla}g + g \vec{\nabla}f$
4. $\vec{\nabla} \left(\frac{f}{g} \right) = \frac{g \vec{\nabla}f - f \vec{\nabla}g}{g^2}$ at points \vec{x} where $g(\vec{x}) \neq 0$
5. $\text{div}(\vec{F} + \vec{G}) = \text{div } \vec{F} + \text{div } \vec{G}$
6. $\overrightarrow{\text{curl}}(\vec{F} + \vec{G}) = \overrightarrow{\text{curl}} \vec{F} + \overrightarrow{\text{curl}} \vec{G}$
7. $\text{div}(f\vec{F}) = f \text{div } \vec{F} + \vec{F} \cdot \vec{\nabla}f$
8. $\text{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \overrightarrow{\text{curl}} \vec{F} - \vec{F} \cdot \overrightarrow{\text{curl}} \vec{G}$
- * 9. $\overrightarrow{\text{curl}} \vec{F} = \vec{0}$ if $\vec{F} \in C^2$
10. $\overrightarrow{\text{curl}}(\vec{F}\vec{F}) = f \overrightarrow{\text{curl}} \vec{F} + \vec{\nabla}f \times \vec{F}$
- * 11. $\overrightarrow{\text{curl}} \vec{\nabla}f = \vec{0}$ if $f \in C^2$
12. $\vec{\nabla}^2(fg) = f \vec{\nabla}^2g + g \vec{\nabla}^2f + 2(\vec{\nabla}f \cdot \vec{\nabla}g)$
13. $\text{div}(\vec{\nabla}f \times \vec{\nabla}g) = 0$ if $f, g \in C^2$
14. $\text{div}(f \vec{\nabla}g - g \vec{\nabla}f) = f \vec{\nabla}^2g - g \vec{\nabla}^2f$

Note: Euler's Theorem **3.1** \implies 9, 11, 13

4.4. Gradient Vector Field

Given: a vector field $\vec{F}: \mathbb{D}^3 \rightarrow \mathbb{R}^3$ where $\mathbb{D}^3 \subset \mathbb{R}^3$

a scalar field $f: \mathbb{D}^3 \rightarrow \mathbb{R}$ and $f \in C^1$

$$\vec{\nabla}f = \vec{F}$$

Then: \vec{F} is called a gradient vector field

f is called the potential of \vec{F} .

Thm. If: $\vec{F}: \mathbb{D}^3 \rightarrow \mathbb{R}^3$ is a vector field

$$\vec{F} \in C^1$$

$$\overrightarrow{\text{curl}} \vec{F} = \vec{0}$$

Then: \vec{F} is NOT a gradient vector field.

Why: by 4.3 #11 above.

4.5. Geometry of the Divergence pg 284-5

4.5a. Given: a VELOCITY vector field $\vec{F}: \mathbb{D}^3 \rightarrow \mathbb{R}^3$ of a compressible fluid

a point $(x_0, y_0, z_0) \in \mathbb{D}^3$ and a small $\varepsilon > 0$

$P(0)$ is the cube (of the fluid) based at (x_0, y_0, z_0) with sides determined by $\varepsilon \vec{i}, \varepsilon \vec{j}, \varepsilon \vec{k}$

$P(t)$ is the (approx.) parallelepiped that $P(0)$ is carried to t time-units later

$V(t)$ = volume of $P(t)$.

Then:

$$\operatorname{div} \vec{F}(x_0, y_0, z_0) = \frac{1}{V(0)} \frac{dV}{dt} \Big|_{(x,y,z)=(x_0,y_0,z_0)}^{t=0}.$$

So: $\operatorname{div} \vec{F}$ is the rate of change of the fluid's volume, per unit vol., under the flow of the fluid

$\operatorname{div} \vec{F}(x_0, y_0, z_0) > 0 \Rightarrow$ the fluid is expanding at $(x_0, y_0, z_0) \Rightarrow (x_0, y_0, z_0)$ is a source

$\operatorname{div} \vec{F}(x_0, y_0, z_0) < 0 \Rightarrow$ the fluid is compressing at $(x_0, y_0, z_0) \Rightarrow (x_0, y_0, z_0)$ is a sink

4.5b. Have: a fluid flowing through a region \mathbb{D}^3 in \mathbb{R}^3 with:

\vec{v} - velocity vector field of the fluid $\left(\frac{\text{m}}{\text{sec}}\right)$

δ - mass density scalar field of the fluid $\left(\frac{\text{g}}{\text{m}^3}\right)$

$\vec{F} \stackrel{\text{def}}{=} \delta \vec{v}$ - mass flow rate density of the fluid $\left(\frac{\text{g}}{\text{sec}} \text{ per m}^2\right)$

a porous surface \mathcal{S} sitting in \mathbb{D}^3 :

through which the fluid flows without hindrance & with outward UNIT normal \vec{n} .

Def: flux of \vec{F} across $\mathcal{S} \stackrel{\text{def}}{=} \frac{\text{net mass of fluid crossing } \mathcal{S} \text{ per unit time } \left(\frac{\text{g}}{\text{sec}}\right)}{\left[\text{mass of fluid coming out of } \mathcal{S}\right] - \left[\text{mass of fluid going into } \mathcal{S}\right]} \text{ during } \Delta t$

Note: $\left[\begin{array}{l} \text{coming out of } \mathcal{S} \\ \text{going into } \mathcal{S} \end{array} \right]$ means the fluid is flowing away from the $\left[\begin{array}{l} \text{positive} \\ \text{negative} \end{array} \right]$ side of \mathcal{S} .

Note: if $\delta \equiv 1$, then flux = the net volume of fluid crossing \mathcal{S} per unit time.

(i) If: \mathcal{S} is a small FLAT patch and \vec{v} and δ are constant on \mathcal{S}
then: flux across $\mathcal{S} = \left(\vec{F} \cdot \vec{n} \right)$ (area of \mathcal{S}).

(ii) If: $\mathcal{S} \equiv \mathcal{S}(\varepsilon)$ is a rectangular box with:
center P_0 , sides \parallel to the coordinate axes, and of volume ε
then: $\mathcal{S}(\varepsilon)$ is the disjoint union of 6 small FLAT patches as above and

$$\operatorname{div} \vec{F}(P_0) = \lim_{\varepsilon \rightarrow 0} \frac{\text{flux of } \vec{F} \text{ across } \mathcal{S}(\varepsilon)}{\text{vol } \mathcal{S}(\varepsilon)}.$$

So: $\operatorname{div} \vec{F}(P_0)$ is the flux of \vec{F} across a small element of volume containing P_0 , per unit volume.

4.6. Rotation pg 278-9

Given: a (slanted/flat) solid rigid body B that is welded to an axis L

the axis L is rotating at a constant angular speed

some point Q on B .

So as L rotates, Q follows a circular path in a plane \perp to L .

Let: \vec{v} = velocity vector field of B

\vec{w} = angular velocity vector of B .

So: $\triangleright \vec{v} \parallel L$

$\triangleright \|\vec{w}\| \equiv \omega \equiv \text{angular speed of } B \equiv \frac{d(\text{angle})}{dt} = \frac{\text{speed of } Q}{d(Q,L)}.$

Then: $\vec{v} = \vec{w} \times \vec{OQ}$
 $\operatorname{curl} \vec{v} = 2\vec{w}$.

4.7. Geometry of the Curl pg 279

Given: a VELOCITY vector field $\vec{F}: \mathbb{D}^3 \rightarrow \mathbb{R}^3$ of a fluid

a point $(x, y, z) \in \mathbb{D}^3$

a RIGID body B centered at (x, y, z) .

Then: $\operatorname{curl} \vec{F}(x, y, z)$ is twice the angular velocity vector of B

as it rotates about its own axis while in the fluid at (x, y, z) .

Def: $\operatorname{curl} \vec{F} = 0 \stackrel{\text{def}}{\iff} \vec{F}$ is IRROTATIONAL.

4.8. Key Idea

Given: a VELOCITY vector field $\vec{F}: \mathbb{D}^3 \rightarrow \mathbb{R}^3$ of a fluid.

Then: $\operatorname{div} \vec{F}$ measures how much the fluid is expanding

$\operatorname{curl} \vec{F}$ measures how much an object is rotating about

its own axis when placed in the fluid.

5.1. Single Integral

Given: a function $f: [a, b] \rightarrow \mathbb{R}$

a partition $P: a = x_0 < x_1 < \dots < x_n = b$ of $[a, b]$

a selection $\{c_i\}_{i=1}^n$ where $c_i \in [x_{i-1}, x_i]$

Defs: $l([x_{i-1}, x_i]) = \text{length } [x_{i-1}, x_i]$

$\text{mesh of } P = |P| = \max_{1 \leq i \leq n} l([x_{i-1}, x_i])$

Riemann sum of $(f, P, \{c_i\}) = \sum_{i=1}^n f(c_i) \cdot l([x_{i-1}, x_i])$

f is integrable over $[a, b]$ with $\int_{[a,b]} f(x) dx = I \stackrel{\text{def}}{=} \lim_{|P| \rightarrow 0} \sum_{i=1}^n f(c_i) \cdot l([x_{i-1}, x_i]) = I$ for each selection of $\{c_i\}$'s.

So: $\int_{[a,b]} 1 dx = l([a, b])$

and: $\sum_{i=1}^n f(c_i) \cdot \frac{l([x_{i-1}, x_i])}{l([a,b])} \approx \text{avg. value of } f \text{ over } [a, b] = \frac{\int_{[a,b]} f(x) dx}{l([a, b])}$.

Def: the area of the region btw. the graph of $y = f(x)$ and $[a, b] \stackrel{\text{def}}{=} \int_{[a,b]} f(x) dx$.

Thm: If f is continuous on $[a, b]$, then f is integrable over $[a, b]$.

Fundamental Theorem of Calculus: If $f = F'$ is continuous, then $\int_{[a,b]} f(x) dx = F(b) - F(a)$.

5.2. Double Integral

Given: $R^2 \subset [a_x, b_x] \times [a_y, b_y] \subset \mathbb{R}^2$

a function $f: R^2 \rightarrow \mathbb{R}$

a partition $P_x: a_x = x_0 < x_1 < \dots < x_n = b_x$ of $[a_x, b_x]$

a partition $P_y: a_y = y_0 < y_1 < \dots < y_m = b_y$ of $[a_y, b_y]$.

Form: rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Pick: a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

Defs: $A(R_{ij}) = \text{area of } R_{ij}$

$\mathcal{R} = \{R_{ij} : R_{ij} \subset R^2\}$

mesh of } \mathcal{R} = |\mathcal{R}| = \max_{i,j} A(R_{ij})

Riemann sum of $(f, P_x, P_y, \{c_{ij}\}) = \sum_{R_{ij} \in \mathcal{R}} f(c_{ij}) \cdot A(R_{ij})$.

f is integrable over R^2 with $\iint_{R^2} f(x, y) dA = I \stackrel{\text{def}}{=} \lim_{|\mathcal{R}| \rightarrow 0} \sum_{R_{ij} \in \mathcal{R}} f(c_{ij}) \cdot A(R_{ij}) = I$ for each selection of $\{c_{ij}\}$'s.

$\iint_{R^2} 1 dA = \text{area of } R^2$

So: $\sum_{R_{ij} \in \mathcal{R}} f(c_{ij}) \cdot \frac{A(R_{ij})}{A(R^2)} \approx \text{avg. value of } f \text{ over } R^2 = \frac{\iint_{R^2} f(x, y) dA}{\text{area of } R^2}$.

Def: the volume of the solid btw. the graph of $z = f(x, y)$ and $R^2 \stackrel{\text{def}}{=} \iint_{R^2} f(x, y) dA$.

Def: R^2 is elementary if, for continuous functions g_i , either:

R^2 is of type 1 = type $_y$ $\stackrel{\text{def}}{\iff} R^2 = \{(x, y) \in \mathbb{R}^2 : g_1(x) \leq y \leq g_2(x) \text{ and } a \leq x \leq b\}$

\iff draw typical rectangle \parallel to y -axis

R^2 is of type 2 = type $_x$ $\stackrel{\text{def}}{\iff} R^2 = \{(x, y) \in \mathbb{R}^2 : g_1(y) \leq x \leq g_2(y) \text{ and } a \leq y \leq b\}$

\iff draw typical rectangle \parallel to x -axis

R^2 is of type 3 = type $_{xy}$ $\stackrel{\text{def}}{\iff} R^2$ is of both type $_x$ and type $_y$.

If: $f: R^2 \rightarrow \mathbb{R}$ is continuous and R^2 is elementary,

then: $\iint_{R^2} f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$ if R^2 is of type $_y$

and: $\iint_{R^2} f(x, y) dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy$ if R^2 is of type $_x$.

5.3. Triple Integral

Given: $R^3 \subset [a_x, b_x] \times [a_y, b_y] \times [a_z, b_z] \subset \mathbb{R}^3$

a function $f: R^3 \rightarrow \mathbb{R}$

a partition $P_x: a_x = x_0 < x_1 < \dots < x_n = b_x$ of $[a_x, b_x]$

a partition $P_y: a_y = y_0 < y_1 < \dots < y_m = b_y$ of $[a_y, b_y]$

a partition $P_z: a_z = z_0 < z_1 < \dots < z_l = b_z$ of $[a_z, b_z]$.

Form: rectangular cubes (ie. boxes) $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$.

Pick: a selection $\{c_{ijk}\}$ where $c_{ijk} \in B_{ijk}$.

Defs: $V(B_{ijk}) = \text{volume of } B_{ijk}$

$\mathcal{V} = \{B_{ijk} : B_{ijk} \subset R^3\}$

mesh of } \mathcal{V} = |\mathcal{V}| = \max_{i,j,k} V(B_{ijk})

Riemann sum of $(f, P_x, P_y, P_z, \{c_{ijk}\}) = \sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot V(B_{ijk})$.

f is integrable over R^3 with $\iiint_{R^3} f(x, y, z) dV = I \stackrel{\text{def}}{\iff}$

$\lim_{|\mathcal{V}| \rightarrow 0} \sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot V(B_{ijk}) = I$ for each selection of $\{c_{ijk}\}$'s.

So: $\iiint_{R^3} 1 dV = \text{volume of } R^3$

and: $\sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot \frac{V(B_{ijk})}{V(R^3)} \approx \text{avg. value of } f \text{ over } R^3 = \frac{\iiint_{R^3} f(x, y, z) dV}{V(R^3)}$

Def: R^3 is z-simple if, $R^3 = \{(x, y, z) \in \mathbb{R}^3 : g_1(x, y) \leq z \leq g_2(x, y) \text{ and } (x, y) \in R^2\}$ where:

$\triangleright g_i \in C^1$, ie. g_i have continuous 1st-order partial derivatives

$\triangleright R^2$ is elementary, as in 5.2.

\rightsquigarrow draw typical box with height \parallel to z -axis

f is continuous and R^3 is z -simple,

then: $\iiint_{R^3} f(x, y, z) dV = \iint_{R^2} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z) dz \right] dA$.

Rmk: Analogs hold for x -simple and y -simple.

6.1. Recall the 1D Change of Variables Formula:

Have: $I^* \xrightarrow{g} I \xrightarrow{f} \mathbb{R}$

where: $I^* = [a, b] \subset \mathbb{R}$

$g \in C^1(I^*)$, ie. g' is continuous on I^*

$I = g(I^*) = [c, d]$

f is continuous .

They: $\underbrace{\int_{g(a)}^{g(b)} f(x) dx}_{\text{get something easy to compute}}$ $\stackrel{(*)}{=} \underbrace{\int_a^b f(g(u)) \frac{dg}{du} du}_{\text{given but hard to compute} \leftrightarrow}$.
so find g and $I^* \leftrightarrow$

usually:

If: $\frac{dg}{du} \geq 0$ on I^* (and so $I = [g(a), g(b)]$)

or: $\frac{dg}{du} \leq 0$ on I^* (and so $I = [g(b), g(a)]$)

Then: formula (*) becomes

$$\int_I f(x) dx \stackrel{(**)}{=} \int_{I^*} f(g(u)) \left| \frac{dg}{du} \right| du .$$

$$\int_I f(x) dx \stackrel{(**)}{=} \int_{I^*} f(x(u)) \left| \frac{dx}{du} \right| du .$$

i.e.: if you prefer, let $g(u) = x(u)$ so

Goal the 2D & 3D Change of Variables Formulas:

Have: $D^* \xrightarrow{T} D \xrightarrow{f} \mathbb{R}$

$D^*, D \subset \mathbb{R}^2$ or 3

$T(D^*) = D$

D^*, D, T, f are nice .

Want: an analog to (**),

like: $\iint_D f(x, y) dA \stackrel{(2D)}{=} \iint_{D^*} f(T(u, v)) \boxed{?} dA$

or: $\iiint_D f(x, y, z) dV \stackrel{(3D)}{=} \iiint_{D^*} f(T(u, v, w)) \boxed{?} dV .$

6.2. Interior & Boundary cf. 5.2 & 5.3

For: the elementary region $R^2 = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$

the interior of R^2 is $\{(x, y) \in \mathbb{R}^2 : a < x < b \text{ and } g_1(x) < y < g_2(x)\}$

the boundary of R^2 is $R^2 \setminus$ [the interior of R^2] .

For: the z -simple region $R^3 = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R^2 \text{ and } g_1(x, y) \leq z \leq g_2(x, y)\}$

the interior of R^3 is $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in$ [the interior of R^2] & $g_1(x, y) < z < g_2(x, y)\}$

the boundary of R^3 is $R^3 \setminus$ [the interior of R^3] .

6.3. One-to-One & Onto & Jacobian

Have: $T: D^* \rightarrow D$

$D^*, D \subset \mathbb{R}^n$

Def: T is one-to-one \Leftrightarrow if $\vec{a}_i \in D^*$ and $\vec{a}_1 \neq \vec{a}_2$ then $T(\vec{a}_1) \neq T(\vec{a}_2)$

\Leftrightarrow if $\vec{a}_i \in D^*$ and $T(\vec{a}_1) = T(\vec{a}_2)$ then $\vec{a}_1 = \vec{a}_2$

$\Leftrightarrow T$ does NOT send two different points of D^* to the same point in D

$\Leftrightarrow T$ does NOT fold-up D

$\stackrel{v=1}{\Leftrightarrow} T$ passes the horizontal line test.

Rmk: in 6.1, if g is 1-to-1, then either $\frac{dg}{du} \geq 0$ or $\frac{dg}{du} \leq 0$ on I^* .

Def: T is onto $\Leftrightarrow T(D^*) = D$

\Leftrightarrow if $\vec{b} \in D$ then there is $\vec{a} \in D^*$ so that $T(\vec{a}) = \vec{b}$

$\Leftrightarrow T$ maps D^* onto all of D .

If: $T \in C^1(D^*)$

then: Jacobian matrix of $T = J_T = D_M T$ (recall D_M was defined in 2.2)

and: Jacobian of $T = \det J_T$.

So : for $n = 2$,

$$T(u, v) = \langle x(u, v), y(u, v) \rangle$$

$$\det J_T = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and for $n = 3$,

$T(u, v, w) = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$

$$\det J_T = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} .$$

To change to cylindrical coordinates : (see 1.8)

let: $x = r \cos \theta$, $y = r \sin \theta$, $z = z$

so: $T(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle$

and: $\det J_T = r$ and $|\det J_T| = r$

To change to spherical coordinates : (see 1.9)

let: $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$

so: $T(\rho, \theta, \phi) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$

and: $\det J_T = -\rho^2 \sin \phi$ and $|\det J_T| = \rho^2 \sin \phi$

6.5. Change of Variables Formula

Let: $D^* \xrightarrow{T} D \xrightarrow{f} \mathbb{R}$
 $D^*, D \subset \left[\frac{\mathbb{R}^2}{\mathbb{R}^3} \right]$ and D^* is [elementary sample] (see [5.2] [5.3])
 $T \in \mathcal{C}^1(D^*)$

T is 1-to-1 on the interior of D^*

$T(D^*) = D$

f is continuous .

For: 2D-case:

$$\iint_D f(x, y) dA = \iint_{D^*} f(T(u, v)) |\det J_T| dA$$

mnemonically:

$$\iint_D f(x, y) \text{ "dx dy" } = \iint_{D^*} f(T(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \text{ "du dv" } .$$

For: 3D-case:

$$\iiint_D f(x, y, z) dV = \iiint_{D^*} f(T(u, v, w)) |\det J_T| dV$$

mnemonically:

$$\iiint_D f(x, y, z) \text{ "dx dy dz" } = \iiint_{D^*} f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \text{ "du dv dw" } .$$

6.4. Examples in \mathbb{R}^2 : $T : D^* \rightarrow D$

6.4.1. Polar Coordinates:

\triangleright $T(r, \theta) = (r \cos \theta, r \sin \theta) \Rightarrow J_T = \underline{\hspace{2cm}} \Rightarrow \det J_T = \underline{\hspace{2cm}}$
a. $D^* = [a, b] \times [0, \pi/2]$ where $0 < a < b \Rightarrow T(D^*) = \underline{\hspace{2cm}}$

b. $D^* = [a, b] \times [0, 3\pi]$ where $0 < a < b \Rightarrow T(D^*) = \underline{\hspace{2cm}}$

c. $D^* = [0, \theta] \times [0, \theta_0]$ where $0 < b$ and $0 < \theta_0 < 2\pi \Rightarrow T(D^*) = \underline{\hspace{2cm}}$

6.4.2. Linear Transformations: here $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

\triangleright $T(u, v) = \mathbf{A} \begin{bmatrix} u \\ v \end{bmatrix} + \vec{b} = \begin{bmatrix} a_{11}u + a_{12}v + b_1 \\ a_{21}u + a_{22}v + b_2 \end{bmatrix} = \begin{bmatrix} x(u, v) \\ y(u, v) \end{bmatrix}$

Facts from linear algebra:

- (i) $\det \mathbf{A} \neq 0 \Leftrightarrow T$ is 1-to-1 (see § 6.1 Ex. 8)
- (ii) $\det \mathbf{A} \neq 0 \Leftrightarrow T(\mathbb{R}^2) = \mathbb{R}^2$ (see § 6.1 Ex. 9)
- (iii) $\mathbf{A}\vec{d}_1 \times \mathbf{A}\vec{d}_2 = (\det \mathbf{A})(\vec{d}_1 \times \vec{d}_2)$ (just write it out)
- (iv) $J_T = \underline{\hspace{2cm}}$.

Consider parallelograms: here $\vec{d}_i \neq \vec{0}$

Let: $D^* = \{ \vec{p} + \lambda \vec{d}_1 + \mu \vec{d}_2 : 0 \leq \lambda, \mu \leq 1 \}$

so: D^* is a parallelogram $\Leftrightarrow \vec{d}_1 \nparallel \vec{d}_2 \Leftrightarrow \vec{d}_1 \times \vec{d}_2 \neq \vec{0}$.

Note: $T(D^*) = \{ (\mathbf{A}\vec{p} + \vec{b}) + \lambda \mathbf{A}\vec{d}_1 + \mu \mathbf{A}\vec{d}_2 : 0 \leq \lambda, \mu \leq 1 \}$.

So: $T(D^*)$ is a parallelogram $\Leftrightarrow D^*$ is a parallelogram and $\underline{\hspace{2cm}}$.

Note: $\mathbf{A}\vec{d} = \langle a_{11}, a_{21} \rangle$ and $\mathbf{A}\vec{j} = \langle a_{12}, a_{22} \rangle$,

so: $T([0, 1] \times [0, 1]) = \{ \vec{b} + \lambda \langle a_{11}, a_{21} \rangle + \mu \langle a_{12}, a_{22} \rangle : 0 \leq \lambda, \mu \leq 1 \}$.

If: D^* & $T(D^*)$ are parallelograms, then

area $T(D^*) = \left\| (\mathbf{A}\vec{d}_1) \times (\mathbf{A}\vec{d}_2) \right\| \stackrel{\text{(iii)}}{=} |\det \mathbf{A}| \left\| \vec{d}_1 \times \vec{d}_2 \right\| = |\det \mathbf{A}| \text{ area } (D^*) \stackrel{\text{(iv)}}{=} |\det J_T| \text{ area } (D^*)$.

7.h. Ch 7 – Summary

- 7.h.1.** Given: a path $[a, b] \xrightarrow{\vec{c}} \mathbb{R}^n$
 a surface $D^2 \xrightarrow[\mathbb{R}^2]{\Phi} \xrightarrow[\mathbb{R}^3]{\text{onto}} \mathcal{S}$
 a scalar field $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$
 a vector field $\mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$
 with: everything nice & continuous.

7.b Path Integral — averages f along the path \vec{c}

$$\int_a^b f \, ds = \int_a^b f(\vec{c}(t)) \|\vec{c}'(t)\| \, dt$$

7.c Line Integral — work done by force \vec{F} on a puffo as he moves along the \vec{c}

$$\int_a^b \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t) \, dt$$

7.e Surface Integral — averages f over the surface \mathcal{S}

$$\iint_{\mathcal{S}} f \, dS = \iint_{D^2} f(\Phi(u, v)) \|\vec{T}_u \times \vec{T}_v\| \, dudv$$

7.g Surface Integral — the flux of \vec{F} across the surface \mathcal{S}

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{D^2} \vec{F} \cdot (\vec{T}_u \times \vec{T}_v) \, dudv = \iint_{\mathcal{S}} [\vec{F} \cdot \vec{n}] \, dS$$

Note: path & line integrals change integrals over paths to single integrals (as in 5.1)
 surface integrals change integrals over surfaces to double integrals (as in 5.2).

7.h.2. Integrals over paths

$$\vec{c}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{c}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$ds = \|\vec{c}'\| \, dt$$

$$d\vec{s} = \vec{c}' \, dt$$

7.h.3. Integrals over Surfaces

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

$$\vec{T}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\vec{T}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$dS = \|\vec{T}_u \times \vec{T}_v\| \, dudv = \sqrt{\left[\frac{\partial(x,y,z)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)} \right]^2} \, dudv$$

$$d\vec{S} = \left(\vec{T}_u \times \vec{T}_v \right) \, dudv = \vec{n} \, dS \quad \text{where } \vec{n} \text{ is the outward unit normal vector to } \mathcal{S}$$

7.a. Paths/Curves

▷ Review 2.6

▷ Given: $[a_1, b_1] \xrightarrow[\text{onto}]^h [a, b] \xrightarrow{\vec{c}} \mathbb{R}^n$
 with: $h \in C^1$ and h is 1-to-1 on $[a_1, b_1]$

\vec{c} is piecewise C^1 on $[a, b]$

then: $\vec{p} \stackrel{\text{def}}{=} \vec{c} \circ h: [a_1, b_1] \rightarrow \mathbb{R}^n$ is a REPARAMETERIZATION of \vec{c} .

Note: $\{\vec{c}(t) : a \leq t \leq b\} = \{\vec{p}(t) : a_1 \leq t \leq b_1\} = \mathcal{C}$.

Since: h is 1-to-1, there are 2 possibilities:

either: $h' > 0$ on $[a_1, b_1] \Rightarrow [a, b] = [h(a_1), h(b_1)] \Rightarrow h$ PRESERVES ORIENTATION

or: $h' < 0$ on $[a_1, b_1] \Rightarrow [a, b] = [h(b_1), h(a_1)] \Rightarrow h$ REVERSES ORIENTATION.

Since: $\vec{p}'(t) = \vec{c}'(h(t))h'(t)$
 one thinks of h as changing the speed at which a point moves along the curve \mathcal{C} .

Ex: If: $[a, b] \xrightarrow[\text{onto}]^h [a, b] \xrightarrow{\vec{c}} \mathbb{R}^n$

with: $h(t) = a + b - t$

then: $\vec{p}(t) = \vec{c}(a + b - t) = \vec{c}_{\text{op}}(t) =$ the opposite path to \vec{c}

= a path in the opposite direction as \vec{c}

but with the same speed as \vec{c} .

Ex: If: $[0, 1] \xrightarrow[\text{onto}]^h [a, b] \xrightarrow{\vec{c}} \mathbb{R}^n$

with: $h(t) = a + (b - a)t$

then: $\vec{p}(t) = \vec{c}(a + (b - a)t)$

= a path in the same direction as \vec{c}

but with the speed of (b-a) times the speed of \vec{c} .

7.b. Paths Integral

7.b.1. Given: $[a, b] \xrightarrow{\vec{c}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

with: $\vec{c} \in C^1[a, b]$

f is a scalar field

f is continuous on $\vec{c}([a, b])$

then: the PATH INTEGRAL of f over \vec{c} is $\boxed{\int_a^b f ds \stackrel{\text{def}}{=} \int_a^b f(\vec{c}(t)) \|\vec{c}'(t)\| dt}$.

If: \vec{c} is piecewise C^1 on $[a, b]$

then: we cope in the obvious way.

7.b.2. Key Idea:

Form: a partition of $[a, b] : a = t_0 < t_1 < \dots < t_n = b$,

with: $\Delta t_i \equiv t_i - t_{i-1} \approx 0$,

pick: a selection $t_i^* \in [t_{i-1}, t_i]$.

Note: $\vec{c}'(t_i^*) \approx [\vec{c}(t_i) - \vec{c}(t_{i-1})] / \Delta t_i$.

Then: $\int_a^b f ds = \int_a^b [f(\vec{c}(t)) \|\vec{c}'(t)\|] dt$
 $\approx \sum_{i=1}^n [f(\vec{c}(t_i^*)) \|\vec{c}'(t_i^*)\|] \Delta t_i$
 $= \sum_{i=1}^n f(\vec{c}(t_i^*)) [\|\vec{c}'(t_i^*)\| \Delta t_i]$
 $\approx \sum_{i=1}^n f(\vec{c}(t_i^*)) \|\vec{c}(t_i) - \vec{c}(t_{i-1})\|$
 $\approx \sum_{i=1}^n f(\vec{c}(t_i^*)) [\text{length of the curve } \vec{c}([t_{i-1}, t_i])].$

7.b.3. Examples:

▷ $\int_a^b 1 ds =$ length of the path \vec{c} .

▷ If: \vec{c} represents a wire in \mathbb{R}^3

with: $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}$ the mass density (eg. grams/cm) of \vec{c}

then: the total mass of the wire $= \int_a^b \rho ds$

and: the center of mass of the wire $= \left(\frac{\int_a^b x \rho ds}{\int_a^b \rho ds}, \frac{\int_a^b y \rho ds}{\int_a^b \rho ds}, \frac{\int_a^b z \rho ds}{\int_a^b \rho ds} \right)$.

▷ the AVERAGE VALUE of f along \vec{c} is $\frac{\int_a^b f ds}{\int_a^b 1 ds}$

eg: $f(x, y, z)$ is the temperature at the point (x, y, z)

so: average value of f along $\vec{c} =$ average temperature of f along \vec{c} .

▷ If: $[a, b] \xrightarrow{\vec{c}} \mathbb{R}^2 \xrightarrow{f} [0, \infty)$ and so $\vec{c}([a, b])$ sits in the xy -plane

then: $\int_a^b f ds$ is the surface area of one side of the fence over \vec{c} with height given by f .

7.b.4. Given:

a reparameterization \vec{p} of \vec{c}

then: $\int_{\vec{p}} f ds = \int_{\vec{c}} f ds$.

7.c. Line Integral

7.c.1. Given: $[a, b] \xrightarrow{\vec{c}} \mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$

with: $\vec{c} \in C^1[a, b]$

\vec{F} is a vector field

\vec{F} is continuous on $\vec{c}([a, b])$

then: the LINE INTEGRAL of \vec{F} over \vec{c} is $\boxed{\int_a^b \vec{F} \cdot d\vec{s} \stackrel{\text{def}}{=} \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t) dt}$.

7.c.2. Key Idea: notation as in **7.b.2** and setting as in **7.c.1**:

Then: $\int_a^b \vec{F} \cdot d\vec{s} = \int_a^b [\vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)] dt$
 $\approx \sum_{i=1}^n [\vec{F}(\vec{c}(t_i^*)) \cdot \vec{c}'(t_i^*)] \Delta t_i$
 $\approx \sum_{i=1}^n \vec{F}(\vec{c}(t_i^*)) \cdot [\vec{c}'(t_i^*) \Delta t_i]$
 $\approx \sum_{i=1}^n \vec{F}(\vec{c}(t_i^*)) \cdot [\vec{c}(t_i) - \vec{c}(t_{i-1})].$

7.c.3. Ex. if: \vec{F} is a force field acting on a puffo

then: $\int_a^b \vec{F} \cdot d\vec{s}$ is the work done by \vec{F} on the puffo as he moves along the path \vec{c} .

7.c.4. Differential Form Notation:

If: $\vec{c}(t) = \langle x(t), y(t), z(t) \rangle$ and $\vec{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$

then: $\int_a^b \vec{F} \cdot d\vec{s} = \int_a^b \langle F_1, F_2, F_3 \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle dt = \int_a^b \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$
notation $\int_{\vec{c}} F_1 dx + F_2 dy + F_3 dz$.

7.c.5. From a Line Integral (**7.c.1** setting) to a Path Integral (**7.b.1** setting).

If: $\vec{c}'(t) \neq \vec{0}$ on $[a, b]$ and \vec{c} is one-to-one on $[a, b]$

so: $\vec{T}(t) \stackrel{\text{def}}{=} \vec{c}'(t) / \|\vec{c}'(t)\|$ is the unit tangent vector to $\vec{c}(t)$

then: $\int_a^b \vec{F} \cdot d\vec{s} = \int_a^b [\vec{F}(\vec{c}(t)) \cdot \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}] \|\vec{c}'(t)\| dt = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{T}(t) \|\vec{c}'(t)\| dt$
 $= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{T}(\vec{c}^{-1}(\vec{c}(t))) \|\vec{c}'(t)\| dt$
 $= \int_{\vec{c}} \vec{F} \cdot (\vec{T} \circ \vec{c}^{-1}) ds$

\rightsquigarrow think of $\vec{F} \cdot (\vec{T} \circ \vec{c}^{-1})$ as the tangential component of \vec{F} along \vec{c} .

7.c.6. Given: the **7.c.1** setting and a reparameterization \vec{p} of \vec{c} .

If \vec{p} preserves orientation then $\int_{\vec{p}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F} \cdot d\vec{s}$.

If \vec{p} reverses orientation then $\int_{\vec{p}} \vec{F} \cdot d\vec{s} = - \int_a^b \vec{F} \cdot d\vec{s}$.

7.c.7. Given: $\vec{c}: [a, b] \rightarrow \mathbb{R}^3$ with \vec{c} piecewise C^1

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ with $f \in C^1$

so $\vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\vec{\nabla} f \in C^1$

then: $\int_{\vec{c}} \vec{\nabla} f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$.

7.d. Surfaces

7.d.1. Given: $\Phi: D^2 \xrightarrow{\text{onto}} \int_{\mathbb{R}^2} S$

so: $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$.

Def: Φ is a PARAMETRIZED SURFACE

S is the SURFACE corresponding to Φ .

If Φ is differentiable (resp. C^1) at $(u, v) \in D^2$

ie. x and y and z are each differentiable (resp. C^1) at $(u, v) \in D^2$

then S is a differentiable (resp. C^1) surface at $\Phi(u, v) \in S$.

If S is a differentiable (resp. C^1) at each $\Phi(u, v) \in S$

then S is a differentiable (resp. C^1) surface.

Rmk: Think of Φ as twisting and bending the 2D region D^2 into a surface S sitting in 3D.

7.d.2. Key Ideas and Defs:

Setup Given: $\Phi: D^2 \xrightarrow{\text{onto}} \int_{\mathbb{R}^3} S$ a parametrized surface

$(u_0, v_0) \in D^2$ with Φ differentiable at (u_0, v_0) ,

so $\Phi(u_0, v_0) \equiv (x_0, y_0, z_0) \in S$.

(A) Then $\vec{c}_{u_0}(t) = \Phi(u_0, t)$ is a path whose image lies on S and goes through (x_0, y_0, z_0) at $t = v_0$

and a tangent vector to \vec{c}_{u_0} at (x_0, y_0, z_0) is: $\vec{T}_v(u_0, v_0) \equiv \vec{c}'_{u_0}(v_0) = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle_{(u_0, v_0)}$

so $\vec{T}_v(u_0, v_0)$ is the instantaneous rate of change in Φ , along $\vec{c}_{u_0}(t)$, at $\Phi(u_0, v_0)$.

Lkws $\vec{c}_{v_0}(t) = \Phi(t, v_0)$ is a path whose image lies on S and goes through (x_0, y_0, z_0) at $t = u_0$

and a tangent vector to \vec{c}_{v_0} at (x_0, y_0, z_0) is: $\vec{T}_u(u_0, v_0) = \vec{c}'_{v_0}(u_0) = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle_{(u_0, v_0)}$

so $\vec{T}_u(u_0, v_0)$ is the instantaneous rate of change in Φ , along $\vec{c}_{v_0}(t)$, at $\Phi(u_0, v_0)$.

↔

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}_{(u_0, v_0)}$$

(B) If $\left[\vec{T}_u \times \vec{T}_v \right] (u_0, v_0) \neq \vec{0}$

then $\left[\vec{T}_u \times \vec{T}_v \right] (u_0, v_0)$ is NORMAL to the surface S at $\Phi(u_0, v_0)$

so an equation of the TANGENT PLANE to the surface S at the point (x_0, y_0, z_0) is:

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \left[\vec{T}_u \times \vec{T}_v \right] (u_0, v_0) = 0.$$

(C) Let: $\left[\vec{T}_u \times \vec{T}_v \right] (u_0, v_0) \neq \vec{0}$

ΔA_0 be the rectangle at (u_0, v_0) , with sides $(\Delta u)\vec{i}$ and $(\Delta v)\vec{j}$.

So $\Phi(\Delta A_0)$ is approximately the parallelogram at $\Phi(u_0, v_0)$ with sides:

$$\Phi(u_0, v_0 + \Delta v) - \Phi(u_0, v_0) = \vec{c}_{u_0}(v_0 + \Delta v) - \vec{c}_{u_0}(v_0) \approx \Delta v \vec{T}_v(u_0, v_0)$$

$$\Phi(u_0 + \Delta u, v_0) - \Phi(u_0, v_0) = \vec{c}_{u_0}(u_0 + \Delta u) - \vec{c}_{u_0}(u_0) \approx \Delta u \vec{T}_u(u_0, v_0)$$

so

$$\text{area } \Delta A_0 = (\Delta u)(\Delta v)$$

$$\text{surface area } \Phi(\Delta A_0) \approx \| T_u(u_0, v_0) \times T_v(u_0, v_0) \| (\Delta u)(\Delta v)$$

$$= \| T_u(u_0, v_0) \times T_v(u_0, v_0) \| (\text{area of } \Delta A_0).$$

(D) If $\left[\vec{T}_u \times \vec{T}_v \right] (u_0, v_0) \neq \vec{0}$

then S is SMOOTH AT $\Phi(u_0, v_0) \in S$.

Also S is SMOOTH if S is smooth at each point of S .

Rmk: loosely speaking, a smooth surface has no corners or breaks.

7.d.3. Have $\Phi: D^2 \xrightarrow{\text{onto}} \int_{\mathbb{R}^3} S$ a parametrized surface

with D^2 is an elementary region in uv -plane

Φ is C^1 and 1-to-1 on the *interior* of D^2

S is smooth, except possibly at a finite number of points of S

then Φ & S are NICE.

If Φ & S are a (finite, disjoint) union of NICE Φ_i 's & S_i 's

then Φ & S are PIECEWISE NICE (pw-nice).

7.d.4. Have $\Phi: D^2 \xrightarrow{\text{onto}} \int_{\mathbb{R}^3} S$ a differentiable parametrized surface,

then

$$\| \vec{T}_u \times \vec{T}_v \| = \sqrt{\left[\frac{\partial(x, y)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(x, z)}{\partial(u, v)} \right]^2}$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

7.d.5. Ex: Special Case

Given $g: D^2 \xrightarrow[\mathbb{R}^3]{\text{onto}} \mathbb{R}$.

Form: $\Phi: D^2 \xrightarrow[\mathbb{R}^3]{\text{onto}} \mathcal{S}$ via $\Phi(x, y) = (x, y, g(x, y))$.

If: D^2 elementary

$g \in C^1$ on interior of D^2

then: Φ is nice

and:

$$\vec{T}_x = \langle 1, 0, \frac{\partial g}{\partial x} \rangle$$

$$\vec{T}_y = \langle 0, 1, \frac{\partial g}{\partial y} \rangle$$

$$\vec{T}_x \times \vec{T}_y = \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle$$

$$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2} + 1$$

and if: $(x_0, y_0) \in D^2$

$P_0 \equiv (x_0, y_0, g(x_0, y_0)) \in \mathcal{S}$

\vec{m} is ANY normal vector to \mathcal{S} at P_0 ,

then:

$$\left\| \vec{T}_x \times \vec{T}_y \right\|_{P_0} = \frac{1}{\left| \cos \angle \vec{k} \vec{m} \right|} = \frac{\|\vec{m}\|}{|\vec{k} \cdot \vec{m}|}$$

7.f. Oriented Surfaces

7.f.1. Consider a parametrized surface $\Phi: D^2 \xrightarrow[\mathbb{R}^3]{\text{onto}} \mathcal{S}$

Intuitively, \mathcal{S} is an ORIENTED SURFACE if it has 2 sides:

- ▷ one side is the outside, or POSITIVE SIDE
- ▷ the other side is the inside, or NEGATIVE SIDE .

At each point $\Phi(u, v) \in \mathcal{S}$, there is a (unique) OUTWARD unit normal vector \vec{n}_{uv} with

- ▷ \vec{n}_{uv} pointing away from the positive side of \mathcal{S}
- ▷ $-\vec{n}_{uv}$ pointing away from the negative side of \mathcal{S} .

We say that $\{\vec{n}_{uv}\}_{(u,v) \in D^2}$ orient \mathcal{S} .

If for each $\Phi(u_0, v_0) \in \mathcal{S}$ where \mathcal{S} is smooth:

- ▷ $\frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} \Big|_{(u_0, v_0)} = \vec{n}_{u_0 v_0}$, then Φ IS ORIENTATION-PRESERVING
- ▷ $\frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} \Big|_{(u_0, v_0)} = -\vec{n}_{u_0 v_0}$, then Φ IS ORIENTATION-REVERSING .

So any parametrized surface for which $\vec{T}_u \times \vec{T}_v$ never vanishes can be considered as an oriented surface with a positive side determined by the direction of the $\vec{T}_u \times \vec{T}_v$'s.

7.f.2. Ex: Sphere $(x^2 + y^2 + z^2 = \rho^2)$

$$\Phi(\theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

$$D^2 = \{(\theta, \phi): 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

$$\vec{T}_\theta \times \vec{T}_\phi = -\langle \rho^2 \sin^2 \phi \cos \theta, \rho^2 \sin^2 \phi \sin \theta, \rho^2 \sin \phi \cos \phi \rangle = -[\rho \sin \phi] \vec{\Phi}(\theta, \phi)$$

$$\|\vec{T}_\theta \times \vec{T}_\phi\| = \rho^2 \sin \phi$$

Φ is orientation-reversing

7.f.3. Ex: Special Case (see **7.d.5** and compare with **2.11**)

$\Phi(x, y) = (x, y, g(x, y))$ where g is differentiable

$$\left[\vec{T}_x \times \vec{T}_y \right] (x_0, y_0) = \frac{\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle}{\left\| \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle \right\|} \Big|_{(x_0, y_0)} \stackrel{\text{def}}{=} \text{the OUTWARD normal unit vector to } \mathcal{S} \text{ at the point } (x_0, y_0, g(x_0, y_0)) \in \mathcal{S}$$

Note: $\left[\vec{T}_x \times \vec{T}_y \right] (x_0, y_0)$ points *upwards* since its last coordinate is positive. Φ is orientation-preserving

7.e. Surface Integrals of Scalar Fields

7.e.1. Given: $D^2 \xrightarrow{\Phi} S \xrightarrow{f} \mathbb{R}$

with: Φ is NICE

f is continuous on $\Phi(D^2)$

then: the INTEGRAL OF f OVER S is:

$$\iint_S f \, dS = \iint_{D^2} f(\Phi(u, v)) \left\| \vec{T}_u \times \vec{T}_v \right\| \, dudv .$$

7.e.2. Key Idea: using notation from 5.2 (see text pg 431&442)

Form: a partition $\mathcal{R} = \{R_{ij} : R_{ij} \subset D^2\}$ of D^2 into rectangles

where: mesh $\mathcal{R} \approx 0$

and: R_{ij} has sides of length Δu_{ij} and Δv_{ij}

pick: a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

$$\begin{aligned} \text{Then: } \iint_S f \, dS &= \iint_{D^2} f(\Phi(u, v)) \left\| \vec{T}_u \times \vec{T}_v \right\| \, dudv \\ &\approx \sum_{R_{ij} \in \mathcal{R}} [f(\Phi(c_{ij})) \left\| \vec{T}_u(c_{ij}) \times \vec{T}_v(c_{ij}) \right\|] \Delta u_{ij} \Delta v_{ij} \\ &= \sum_{R_{ij} \in \mathcal{R}} f(\Phi(c_{ij})) \left[\left\| \vec{T}_u(c_{ij}) \times \vec{T}_v(c_{ij}) \right\| \Delta u_{ij} \Delta v_{ij} \right] \\ &\stackrel{7.d.2}{\approx} \sum_{R_{ij} \in \mathcal{R}} f(\Phi(c_{ij})) [\text{surface area of } \Phi(R_{ij})] . \end{aligned}$$

7.e.3. Examples: compare with 7.b.3

▷ $\iint_S 1 \, dS =$ surface area of S .

▷ If: S represents a surface in \mathbb{R}^3

with: $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}$ the mass density (eg. grams/cm²) of S

then: the total mass of the $S = \iint_S \rho \, dS$

and: the center of mass of $S = \left(\frac{\iint_S x \rho \, dS}{\iint_S \rho \, dS}, \frac{\iint_S y \rho \, dS}{\iint_S \rho \, dS}, \frac{\iint_S z \rho \, dS}{\iint_S \rho \, dS} \right)$.

▷ the AVERAGE VALUE of f on S is $\frac{\iint_S f \, dS}{\iint_S 1 \, dS}$

eg: $f(x, y, z)$ is the temperature at the point (x, y, z)

so: average value of f on $S =$ average temperature of f on S .

7.g. Surface Integrals of Vector Fields

7.g.1. Given $D^2 \xrightarrow{\Phi} S \xrightarrow{\vec{F}} \mathbb{R}^3$

where Φ is NICE and \vec{F} is continuous on $\Phi(D^2)$

and S is oriented by Φ , i.e.

$$\frac{(\vec{T}_u \times \vec{T}_v)}{\left\| \vec{T}_u \times \vec{T}_v \right\|} = \text{the OUTWARD unit normal vector to } S \stackrel{\text{def}}{=} \vec{n} .$$

Then the SURFACE INTEGRAL OF \vec{F} OVER S is:

$$\iint_S \vec{F} \cdot d\vec{S} \stackrel{\text{def}}{=} \iint_{D^2} \left[\vec{F}(\Phi(u, v)) \cdot (\vec{T}_u \times \vec{T}_v) \right] \, dudv \stackrel{7.e.1}{=} \iint_S \left[\vec{F} \cdot \vec{n} \right] \, dS .$$

7.g.2. In the set-up of 7.g.1 and 7.e.1:

If $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ is a $\begin{bmatrix} \text{orientation-preserving} \\ \text{orientation-reversing} \end{bmatrix}$ (re-)parameterization of S

then $\vec{n} = \frac{(\vec{T}_u \times \vec{T}_v)}{\left\| \vec{T}_u \times \vec{T}_v \right\|}$ for $\Phi = \frac{(\vec{T}_u \times \vec{T}_v)}{\left\| \vec{T}_u \times \vec{T}_v \right\|}$ for $\Phi_1 = -\frac{(\vec{T}_u \times \vec{T}_v)}{\left\| \vec{T}_u \times \vec{T}_v \right\|}$ for Φ_2

so $\vec{T}_u \times \vec{T}_v$ for $\Phi = \vec{T}_u \times \vec{T}_v$ for $\Phi_1 = -(\vec{T}_u \times \vec{T}_v)$ for Φ_2

so $\iint_{\Phi} \vec{F} \cdot d\vec{S} = \iint_{\Phi_1} \vec{F} \cdot d\vec{S} = -\iint_{\Phi_2} \vec{F} \cdot d\vec{S}$

and $\iint_{\Phi} f \, dS = \iint_{\Phi_1} f \, dS = \iint_{\Phi_2} f \, dS$

7.g.3. Key Idea: using notation from 7.e.2 (see text pg 431 & 455)

$$\begin{aligned} \text{So: } \iint_S \vec{F} \cdot d\vec{S} &\approx \sum_{R_{ij} \in \mathcal{R}} \left[\vec{F}(\Phi(c_{ij})) \cdot (\vec{T}_u(c_{ij}) \times \vec{T}_v(c_{ij})) \right] \Delta u_{ij} \Delta v_{ij} \\ &= \sum_{R_{ij} \in \mathcal{R}} \left[\vec{F}(\Phi(c_{ij})) \cdot \vec{n}(\Phi(c_{ij})) \right] \left(\left\| \vec{T}_u(c_{ij}) \times \vec{T}_v(c_{ij}) \right\| \Delta u_{ij} \Delta v_{ij} \right) \\ &\stackrel{7.d.2}{\approx} \sum_{R_{ij} \in \mathcal{R}} \left[\vec{F}(\Phi(c_{ij})) \cdot \vec{n}(\Phi(c_{ij})) \right] [\text{surface area of } \Phi(R_{ij})] . \end{aligned}$$

7.g.4. FLUX Recall 4.5b

Have: a fluid flowing through a region \mathbb{D}^3 in \mathbb{R}^3 with:

\vec{v} - velocity vector field of the fluid $\left(\frac{\text{m}}{\text{sec}} \right)$

δ - mass density scalar field of the fluid $\left(\frac{\text{kg}}{\text{m}^3} \right)$

$\vec{F} \stackrel{\text{def}}{=} \delta \vec{v}$ - mass flow rate density of the fluid $\left(\frac{\text{kg}}{\text{sec}} \text{ per m}^2 \right)$

a porous surface S sitting in \mathbb{D}^3 :

through which the fluid flows without hindrance & with outward UNIT normal \vec{n} .

Def: flux of \vec{F} across $S \stackrel{\text{def}}{=} \text{net mass of fluid crossing } S \text{ per unit time}$

$$= \frac{(\text{mass of fluid coming out of } S) - (\text{mass of fluid going into } S)}{\Delta t}$$

Claim: flux of \vec{F} across $S = \iint_S \left[\vec{F} \cdot \vec{n} \right] \, dS$

8.a. Summary of Ch 8 $\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$; $\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$; $\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$

Example: $D \stackrel{\text{def}}{=} \{(x, y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ is a Green's Region
 $\delta D = \{(x, y) : x^2 + y^2 = 1\} \subset D$ is the boundary of D
 $\delta D^+ = \langle \cos t, \sin t \rangle$ $0 \leq t \leq 2\pi$ is a (counterclockwise) parameterization of δD
 $\vec{n} = \langle x, y \rangle$ is the outward unit normal to δD .

8.a.1. Let: D be a Green's region
 $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be C^1 on D .

► Green's Theorem for $D \subset \mathbb{R}^2$

$$\text{Then: } \int_{\delta D^+} \vec{F} \cdot d\vec{s} = \iint_D [\text{curl } \vec{F} \cdot \vec{k}] dA$$

$$\parallel$$

$$\text{ie: } \int_{\delta D^+} P dx + Q dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

$$\rightsquigarrow \text{so: } \int_{\delta D^+} \langle -y, x \rangle \cdot d\vec{s} = 2 \text{ (area of } D \text{)}.$$

► Divergence Theorem for $D \subset \mathbb{R}^2$

$$\text{Then: } \int_{\delta D^+} [\vec{F} \cdot \vec{n}] ds = \iint_D [\text{div } \vec{F}] dA.$$

Example: $S \stackrel{\text{def}}{=} \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\} \subset \mathbb{R}^3$ is a Stokes' Surface
 $\delta S = \{(x, y, 0) : x^2 + y^2 = 1\} \subset S$ is the boundary of S
 $\delta S^+ = \langle \cos t, \sin t, 0 \rangle$ $0 \leq t \leq 2\pi$ is a (cc-wise) parameterization of δS .

8.a.2. Stokes' Theorem for $S \subset \mathbb{R}^3$

If: S is a Stokes' Surface
 $\vec{F} : S \rightarrow \mathbb{R}^3$ is C^1 on S

$$\text{then: } \int_{\delta S^+} \vec{F} \cdot d\vec{s} = \iint_S [\text{curl } \vec{F}] \cdot d\vec{S}.$$

Example: $\Omega \stackrel{\text{def}}{=} \text{ soup can (top \& bottom included) + the soup}$ is a Gauss' Region
 $\delta\Omega = \text{ the soup can (top \& bottom included)}$ is the boundary of Ω .

8.a.3. Gauss' Divergence Theorem for $\Omega \subset \mathbb{R}^3$

If: Ω is a Gauss' Region
 $\vec{F} : \Omega \rightarrow \mathbb{R}^3$ is C^1 on Ω

$$\text{then: } \iint_{\delta\Omega} \vec{F} \cdot d\vec{S} = \iiint_{\Omega} [\text{div } \vec{F}] dV.$$

8.b. Definitions of Green's Region, Stokes' Surface, Gauss' Region
8.b.1. Elementary Green's Region D and corresponding items:

$D \subset \mathbb{R}^2$ is an elementary region (5.2)
 δD is the boundary of D (6.2)
 $\delta D^+ = \langle x(t), y(t) \rangle$ is a piecewise-smooth (oriented simple closed) path that parameterizes δD in the positive (ie. counterclockwise) direction
 $\vec{n} = \frac{\langle y'(t), -x'(t) \rangle}{\| \langle y'(t), -x'(t) \rangle \|}$ is the outward unit normal vector to δD .

\rightsquigarrow note that if you walk along δD^+ , then D is to your left
 \rightsquigarrow a region that can be divided into a finite number of Elementary Green's Regions, with their shared δD 's "oppositely oriented," is a Green's Region (see p. 471)
 Elementary Stokes' Surface S and corresponding items:

Have: $D^2 \xrightarrow{\Phi} S$
 $\int_{\mathbb{R}^2} \int_{\mathbb{R}^3}$
 where Φ is a NICE parameterization of the surface S (7.d.3) & D is an elem. Green's region
 Φ is C^2 on (all of) D and 1-to-1 on (all of) D
 S is smooth and oriented by Φ . (7.f.1)
 So have: $\delta D^+ \stackrel{\text{def}}{=} \langle u(t), v(t) \rangle$ is as above in 8.b.1.
 Let: $\delta S \stackrel{\text{def}}{=} \Phi(\delta D)$
 $\delta S^+ \stackrel{\text{def}}{=} \Phi(\delta D^+)$.

So: δS is the boundary of S

δS^+ is a (oriented simple closed) path that parameterizes δS .

also need: if you walk along δS^+ , standing on the positive side of S , then S is to your left.

\rightsquigarrow Special Case+ $\Phi(x, y) = (x, y, g(x, y))$ where $z = g(x, y)$ is C^2 and D is an elem. Green's region.
 Think of $D \subset \mathbb{R}^2$ as a piece of lycra with a wire as its boundary and Φ places D down in \mathbb{R}^3 and then gently transforms it (1-to-1-ish) to S so that S 's outward unit normals vary continuously.

\rightsquigarrow a surface that can be divided into a finite number of Elementary Stokes' Surfaces, with their shared δS 's "oppositely oriented," is a Stokes' Surface
 Gauss' Region Ω and corresponding items:

$\Omega \subset \mathbb{R}^3$ is a simple region (5.3), thus has top, bottom, and possibly sides
 $\delta\Omega \subset \mathbb{R}^3$ is the boundary of Ω (6.2)
 $\delta\Omega$ is oriented by the outward normal vectors
 $\delta\Omega$ is piecewise NICE (7.d.3).

STOKE'S SURFACES

8.d.1. Let: $D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$.

Define $\Phi: D^2 \xrightarrow{\text{onto}} S$ by

$$\Phi(x, y) = \left\langle \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1} \right\rangle.$$

Claim: $\Phi(x, y)$ is the pt. of intersection of S & the line thru $(x, y, 0)$ & $(0, 0, -1)$.

Note

$$\vec{T}_x \times \vec{T}_y = \left\langle \frac{8x}{(x^2 + y^2 + 1)^3}, \frac{8y}{(x^2 + y^2 + 1)^3}, \frac{4(1 - x^2 - y^2)}{(x^2 + y^2 + 1)^3} \right\rangle.$$

Thus $\Phi: D^2 \xrightarrow{\text{onto}} S$ realizes the upper-hemisphere as a Stoke's surface.

Also: $\delta D^+(t) = \langle \cos t, \sin t \rangle$

$\delta S^+(t) \stackrel{\text{def}}{=} \Phi(\delta D^+(t)) = \langle \cos t, \sin t, 0 \rangle$

So we now see that the sphere is a Stoke's Surface (why?). What's $\delta S^+(t)$?

8.d.2. Let: $0 < h = \text{height}$ and $0 < r = \text{radius}$.

$D^2 = \{(x, y) : -r \leq x \leq r \text{ and } 0 \leq y \leq h\}$

$S = \{(x, y, z) : x^2 + z^2 = r^2 \text{ and } z \geq 0 \text{ and } 0 \leq y \leq h\}$.

Define $\Phi: D^2 \xrightarrow{\text{onto}} S$ by

$$\Phi(x, y) = \left\langle \frac{2xy^2}{r^2 + x^2}, y, \frac{r(r^2 - x^2)}{r^2 + x^2} \right\rangle.$$

Claim: $\Phi(x, y)$ is the pt. of intersection of S & the line thru $(x, y, 0)$ & $(0, y, -r)$.

Note

$$\vec{T}_x \times \vec{T}_y = \left\langle \frac{4xy^4}{(r^2 + x^2)^2}, 0, \frac{2r^2(r^2 - x^2)}{(r^2 + x^2)^2} \right\rangle.$$

Thus $\Phi: D^2 \xrightarrow{\text{onto}} S$ realizes a semi-cylinder as a Stoke's surface.

Also: $\delta D^+(t) = ?$

$\delta S^+(t) \stackrel{\text{def}}{=} \Phi(\delta D^+(t)) = ?$

So we now see that a cylinder is a Stoke's Surface (why?). What's $\delta S^+(t)$?

GAUSS' THEOREM

The key idea behind the proof of Gauss' Theorem is **4.5b**.

To see this, first recall **4.5b**; if you wish, let $\delta = 1$ so the flux is just the net volume of fluid crossing a surface per unit time.

8.e.

Have: a fluid flowing through a region \mathbb{D}^3 in \mathbb{R}^3 with:

- \vec{v} - velocity vector field of the fluid $\left(\frac{\text{m}}{\text{sec}}\right)$
- δ - mass density scalar field of the fluid $\left(\frac{\text{m}}{\text{m}^3}\right)$
- $\vec{F} \stackrel{\text{def}}{=} \delta \vec{v}$ - mass flow rate density of the fluid $\left(\frac{\text{m}}{\text{sec}} \text{ per m}^2\right)$

a porous GAUSS REGION Ω sitting in \mathbb{D}^3 so that:

- the fluid flows through Ω without hindrance
- $\delta\Omega$ is a surface with outward UNIT normal \vec{n} .

8.e.1. Divide $\delta\Omega$ into lots of small (almost flat) patches S_i .

4.5.b (i) \implies flux of \vec{F} across $S_i \approx (\vec{F} \cdot \vec{n})$ (area S_i).

Calculus \implies flux of \vec{F} across $\delta\Omega = \iint_{\delta\Omega} (\vec{F} \cdot \vec{n}) \, dS = \iint_{\delta\Omega} \vec{F} \cdot d\vec{S}$.

8.e.2. Divide Ω into lots of small (solid) rectangular boxes R_i .

4.5.b (ii) \implies flux of \vec{F} across $\delta R_i \approx (\text{div } \vec{F})$ (vol R_i).

Calculus \implies flux of \vec{F} across $\delta\Omega = \iiint_{\Omega} [\text{div } \vec{F}] \, dV$.

8.c. Conservative Fields

8.c.1. Theorem (compare with 4.4)

If $\vec{F}: \mathbb{D}^3 \rightarrow \mathbb{R}^3$ is a C^1 vector field

where $\mathbb{D}^3 =$ all of \mathbb{R}^3 expect for possibly a finite number of points,

then the following are equivalent:

(1) $\vec{F} = \vec{\nabla} f$ for some scalar function $f: \mathbb{D}^3 \rightarrow \mathbb{R}$

(2) $\overrightarrow{\text{curl}} \vec{F} = \vec{0}$

(3) $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = 0$ for each oriented simple closed curve \vec{c}

(4) $\int_{\vec{c}_1} \vec{F} \cdot d\vec{s} = \int_{\vec{c}_2} \vec{F} \cdot d\vec{s}$ for each pair of oriented simple curve \vec{c}_1 & \vec{c}_2

that both start at the same point and both end at the same point.

If \vec{F} satisfies one (and hence all) of the above conditions,

then: \vec{F} is a GRADIENT VECTOR FIELD

\vec{F} is a CONSERVATIVE VECTOR FIELD

f is the POTENTIAL of \vec{F}

$$f(x, y, z) = \int_0^x F_1(t, 0, 0) dt + \int_0^y F_2(x, t, 0) dt + \int_0^z F_3(x, y, t) dt$$

where $\vec{F} = (F_1, F_2, F_3)$.

7.c.7. Given: $\vec{c}: [a, b] \rightarrow \mathbb{R}^3$

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$

so $\vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

then: $\int_{\vec{c}} \vec{\nabla} f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$.

with \vec{c} piecewise C^1

with $f \in C^2$

and $\vec{\nabla} f \in C^1$