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Math 141 HANDOUT

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Fundamental Theorem of Calculus (FTC)

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function.

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- If  $F$  is an antiderivative of  $f$  on  $[a, b]$  (i.e.  $F'(x) = f(x)$  for each  $x \in [a, b]$ ), then

$$\int_a^b f(x) dx \equiv \int_a^b F'(x) dx = F(b) - F(a) .$$

- If  $F(x) = \int_a^x f(t) dt$  for each  $x \in [a, b]$ , then  $F$  is an antiderivative of  $f$  on  $[a, b]$ , i.e.

$$F'(x) \equiv D_x \left[ \int_a^x f(t) dt \right] = f(x) .$$

Basic Differentiation Rules

If the functions  $y = f(x)$  and  $y = g(x)$  are differentiable at  $x$  and  $a$  and  $b$  are constants, then:

1.  $D_x [af(x) + bg(x)] = af'(x) + bg'(x)$
2.  $D_x [f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
3.  $D_x \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$  provided  $g(x) \neq 0$
4.  $D_x [f(x)]^r = r [f(x)]^{r-1} f'(x)$  provided  $r \in \mathbb{Q}$

If  $f$  is differentiable at  $x$  and  $g$  is differentiable at  $f(x)$ , then:

5.  $D_x [g(f(x))] = g'(f(x)) f'(x)$

Generalized Exponential and Logarithmic Functions with base  $a$  where  $a > 0$  but  $a \neq 1$

DERIVATIVES	$\xrightarrow{\text{FTC}}$	INTEGRALS
$D_x \ln  u  \stackrel{u \neq 0}{=} \frac{1}{u} \frac{du}{dx}$		$\int \frac{du}{u} \stackrel{u \neq 0}{=} \ln  u  + C$
$D_x e^u = e^u \frac{du}{dx}$		$\int e^u du = e^u + C$
$D_x \log_a  u  \stackrel{u \neq 0}{=} \frac{1}{u} \frac{1}{\ln a} \frac{du}{dx}$		$\int a^u du = \frac{a^u}{\ln a} + C$

## TRIG and CALCULUS

DERIVATIVES	$\xrightarrow{\text{FTC}}$	INTEGRALS
$D_x \sin u = \cos u \frac{du}{dx}$		$\int \cos u \, du = \sin u + C$
$D_x \tan u = \sec^2 u \frac{du}{dx}$		$\int \sec^2 u \, du = \tan u + C$
$D_x \sec u = \sec u \tan u \frac{du}{dx}$		$\int \sec u \tan u \, du = \sec u + C$
$D_x \cos u = -\sin u \frac{du}{dx}$		$\int \sin u \, du = -\cos u + C$
$D_x \cot u = -\csc^2 u \frac{du}{dx}$		$\int \csc^2 u \, du = -\cot u + C$
$D_x \csc u = -\csc u \cot u \frac{du}{dx}$		$\int \csc u \cot u \, du = -\csc u + C$

### MORE INTEGRALS

$$\begin{aligned}
 \int \tan u \, du &= -\ln |\cos u| + C &= \ln |\sec u| + C \\
 \int \cot u \, du &= \ln |\sin u| + C &= -\ln |\csc u| + C \\
 \int \sec u \, du &= \ln |\sec u + \tan u| + C &= -\ln |\sec u - \tan u| + C \\
 \int \csc u \, du &= -\ln |\csc u + \cot u| + C &= \ln |\csc u - \cot u| + C
 \end{aligned}$$

DERIVATIVES	$\xrightarrow{\text{FTC}}$	INTEGRALS ( $a > 0$ )
$D_x \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$		$\int \frac{1}{\sqrt{a^2-u^2}} \, du = \sin^{-1} \frac{u}{a} + C$
$D_x \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$		$\int \frac{1}{a^2+u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
$D_x \sec^{-1} u = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$		$\int \frac{1}{u\sqrt{u^2-a^2}} \, du = \frac{1}{a} \sec^{-1} \frac{ u }{a} + C$
$D_x \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$		
$D_x \cot^{-1} u = \frac{-1}{1+u^2} \frac{du}{dx}$		
$D_x \csc^{-1} u = \frac{-1}{ u \sqrt{u^2-1}} \frac{du}{dx}$		

Natural Logarithm Fn. $y = \ln x$	AND	Natural Exponential Fn. $y = \exp x$
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$\ln x \stackrel{x \geq 0}{\equiv} \int_1^x \frac{dt}{t}$		nat. exp. fn. $\equiv$ inverse of the nat. log. fn.
$\ln: (0, \infty) \rightarrow (-\infty, \infty)$		$\exp: (-\infty, \infty) \rightarrow (0, \infty)$
$y = \ln x$	$\iff$	$x = \exp y$
$\exists$ a unique $e \in \mathbb{R}$ so that $\ln e = 1$		$e^x \stackrel{x \in \mathbb{R}}{\equiv} \exp(x)$

$$\underline{x, y > 0 \ \& \ r \in \mathbb{Q}}$$

$$e^{\ln x} = x$$

$$\ln 1 = 0$$

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^r) = r(\ln x)$$

$$\underline{x, y \in \mathbb{R} \ \& \ r \in \mathbb{Q}}$$

$$\ln(e^x) = x$$

$$e^0 = 1$$

$$e^x e^y = e^{x+y}$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$(e^x)^r = e^{xr}$$

Generalized Exponential $y = a^x$ and Logarithmic $y = \log_a x$ Functions
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with base $a$ where $a > 0$ but $a \neq 1$ and $b > 0$ but $b \neq 1$
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$\log_e \equiv \ln$
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$$f(x) = a^x \equiv e^{x \ln a} \quad : (-\infty, \infty) \rightarrow (0, \infty)$$

$$g(x) = \log_a x \equiv \text{the inverse of the fn. } f(x) = a^x \quad : (0, \infty) \rightarrow (-\infty, \infty)$$

$$y = \log_a x \quad \iff \quad x = a^y$$

$$(\log_a b)(\log_b c) = \log_a c \quad \implies \quad \log_a x = \frac{\ln x}{\ln a}$$

$$\underline{x, y > 0 \ \& \ r \in \mathbb{R}}$$

$$a^{\log_a x} = x$$

$$\log_a 1 = 0$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^r) = r(\log_a x)$$

$$\underline{x, y \in \mathbb{R} \ \& \ r \in \mathbb{R} \ \& \ 0 < b \neq 1}$$

$$\log_a(a^x) = x$$

$$a^0 = 1$$

$$a^x a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^r = a^{xr}$$

$$(ab)^x = a^x b^x \quad \text{and} \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Basic Trig

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \qquad \sin \theta = \frac{\text{opp}}{\text{hyp}} \qquad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Basic Inverse Trig Functions

$y = \sin \theta$	$\Leftrightarrow$	$\theta = \sin^{-1} y$	where	$-1 \leq y \leq 1$	and	$\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$y = \cos \theta$	$\Leftrightarrow$	$\theta = \cos^{-1} y$	where	$-1 \leq y \leq 1$	and	$0 \leq \theta \leq \pi$
$y = \tan \theta$	$\Leftrightarrow$	$\theta = \tan^{-1} y$	where	$y \in \mathbb{R}$	and	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$y = \cot \theta$	$\Leftrightarrow$	$\theta = \cot^{-1} y$	where	$y \in \mathbb{R}$	and	$0 < \theta < \pi$
$y = \sec \theta$	$\Leftrightarrow$	$\theta = \sec^{-1} y$	where	$ y  \geq 1$	and	$0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$
$y = \csc \theta$	$\Leftrightarrow$	$\theta = \csc^{-1} y$	where	$ y  \geq 1$	and	$\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0$

Math 142

Integration by Parts:  $\int u dv = uv - \int v du$

Trig Identities: (\*) do not have to memorize but should be able to use if given

$\cos(2x) = \cos^2 x - \sin^2 x$	$\sin(2x) = 2 \sin x \cos x$
$\cos^2 x = \frac{1 + \cos(2x)}{2}$	$\sin^2 x = \frac{1 - \cos(2x)}{2}$
$\cos(s+t) \stackrel{(*)}{=} \cos s \cos t - \sin s \sin t$	$\sin(s+t) \stackrel{(*)}{=} \sin s \cos t + \cos s \sin t$
$\cos(s-t) \stackrel{(*)}{=} \cos s \cos t + \sin s \sin t$	$\sin(s-t) \stackrel{(*)}{=} \sin s \cos t - \cos s \sin t$

Trig Substitution

IF INTEGRAND INVOLVES	THEN MAKE THE SUBSTITUTION	RESTRICTION ON $\theta$
$a^2 - u^2$	$u = a \sin \theta \iff \theta = \sin^{-1} \frac{u}{a}$	$\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$a^2 + u^2$	$u = a \tan \theta \iff \theta = \tan^{-1} \frac{u}{a}$	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$u^2 - a^2$	$u = a \sec \theta \iff \theta = \sec^{-1} \frac{u}{a}$	$0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$