SOME MORE WEAK HILBERT SPACES

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Abstract: We give new examples of weak Hilbert spaces.

1. Introduction

The Banach space properties **weak type 2** and **weak cotype 2** were introduced and studied by V. Milman and G. Pisier [MP]. Later, Pisier [P] studied spaces which are both of weak type 2 and weak cotype 2 and called them **weak Hilbert spaces**. Weak Hilbert spaces are stable under passing to subspaces, dual spaces, and quotient spaces. The canonical example of a weak Hilbert space which is not a Hilbert space is convexified Tsirelson space $T^{(2)}$ [CS, J1, J2, P]. Tsirelson's space was introduced by B.S. Tsirelson [T] as the first example of a Banach space which does not contain an isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$. Today, we denote by T the dual space of the original example of Tsirelson since in T we have an important analytic description of the norm due to Figiel and Johnson [FJ]. In [J1], Johnson introduced **modified Tsirelson space** T_M. Later, Casazza and Odell [CO] proved the surprising fact that T_M is naturally isomorphic to the original Tsirelson space T. At this point, all the non-trivial examples of weak Hilbert spaces (i.e. those which are not Hilbert spaces) had unconditional bases and had subspaces which failed to contain ℓ_2. A. Edgington [E] introduced a class of weak Hilbert spaces with unconditional bases which are ℓ_2-saturated. That is, every subspace of the space contains a further subspace isomorphic to a Hilbert space but the space itself is not isomorphic to a Hilbert space. R. Komorowski [K] (or more generally Komorowski and Tomczak-Jaegermann [KT]) proved that there are weak

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Hilbert spaces with no unconditional basis. In fact, they show that $T^{(2)}$ has such subspaces. In another surprise, Nielsen and Tomczak-Jaegermann [NTJ] showed that all weak Hilbert spaces with unconditional bases are very much like $T^{(2)}$.

There are still many open questions concerning weak Hilbert spaces and $T^{(2)}$, due partly to the shortage of non-trivial examples in this area. For example, it is still a major open question in the field whether a Banach space for which every subspace has an unconditional basis (or just local unconditional structure - LUST) must be isomorphic to a Hilbert space. If there are such examples, they will probably come from the class of weak Hilbert spaces. It is an open question whether every weak Hilbert space has a basis, although Maurey and Pisier (see [M]) showed that separable weak Hilbert spaces have finite dimensional decompositions. Nielsen and Tomczak-Jaegermann have shown that weak Hilbert spaces that are Banach lattices have the property that every subspace of every quotient space has a basis. But it is unknown whether every weak Hilbert space can be embedded into a weak Hilbert space with an unconditional basis. In fact, it is unknown if a weak Hilbert space embeds into a Banach lattice of finite cotype. It turns out that this question is equivalent to the question of whether every subspace of a weak Hilbert space must have the GL-Property [CN] which is slightly weaker than having LUST. In this note we extend the list of non-trivial examples of weak Hilbert spaces by producing examples which are ℓ_2-saturated but not isomorphic to the previously known examples.

2. Basic Constructions

If F is a finite dimensional Banach space then let $d(F)$ denote the Banach-Mazur distance between F and $\ell_2^{\dim F}$. The fundamental notion of this note is the one of the weak Hilbert space. Recall the following definition as one of the many equivalent ones (cf [P] Theorem 2.1).

Definition 2.1. *A Banach space X is said to be a weak Hilbert space if there exist $\delta > 0$ and $C \geq 1$ such that for every finite dimensional subspace E of X there exists a subspace $F \subseteq E$ and a projection $P : X \to F$ such that* $\dim F \geq \delta \dim E$, $d(F) \leq C$ *and* $\|P\| \leq C$.

We need to recall the definition of the Schreier sets S_n, $n \in \mathbf{N}$ [AA]. For $F, G \subset \mathbf{N}$, we write $F < G$ (resp. $F \leq G$) when $\max(F) < \min(G)$ (resp. $\max(F) \leq \min(G)$) or one of them is empty, and we write $n \leq F$ instead of $\{n\} \leq F$.

$$S_0 = \{\{n\} : n \in \mathbf{N}\} \cup \{\emptyset\}.$$

If $n \in \mathbf{N} \cup \{0\}$ and S_n has been defined,

$$S_{n+1} = \{\cup_1^n F_i : n \in \mathbf{N}, \mathrm{n} \leq \mathrm{F}_1 < \mathrm{F}_2 < \cdots < \mathrm{F}_\mathrm{n} \text{ and } \mathrm{F}_\mathrm{i} \in \mathrm{S}_\mathrm{n} \text{ for } 1 \leq \mathrm{i} \leq \mathrm{n}\}.$$

For $n \in \mathbf{N}$ a family of finite non-empty subsets (E_i) of $\mathbf{N}$ is said to be S_n*-admissible* if $E_1 < E_2 < \cdots$ and $(\min(E_i)) \in S_n$. Also, (E_i) is said to be S_n*-allowable* if $E_i \cap E_j = \emptyset$ for $i \neq j$ and $(\min(E_i)) \in S_n$.

Every Banach space with a basis can be viewed as the completion of c_{00} (the linear space of finitely supported real valued sequences) under a certain norm. (e_i) will denote the unit vector basis for c_{00} and whenever a Banach space $(X, \|\cdot\|)$ with a basis is regarded as the completion of $(c_{00}, \|\cdot\|)$, (e_i) will denote this (normalized) basis. If $x \in c_{00}$ and $E \subseteq \mathbf{N}$, $Ex \in c_{00}$ is the restriction of x to E; $(Ex)_j = x_j$ if $j \in E$ and 0 otherwise. Also the *support* of x, $\mathrm{supp}\,(x)$, (w.r.t. (e_i)) is the set $\{j \in \mathbf{N} : \mathrm{x(j)} \neq 0\}$. We say that a vector $x \in c_{00}$ is *supported after* n in $n \leq \mathrm{supp}\,(x)$. If $f : \mathbf{R} \to \mathbf{R}$ is a function with $f(0) = 0$ then for $x \in c_{00}$ $f(x)$ will denote the vector $f(x) = (f(x_i))$ in c_{00}.

Let $(X, \|.\|)$ be a Banach space with an unconditional basis. *Whenever a Banach space is considered in this paper we assume that it has an unconditional basis.* The norm of X is 2-convex provided that

$$\|(x^2 + y^2)^{1/2}\| \leq (\|x\|^2 + \|y\|^2)^{1/2}$$

for all vectors $x, y \in X$. The 2-convexification of $(X, \|.\|)$ is the Banach space $(X^{(2)}, \|.\|_{(2)})$ with an unconditional basis, where $x \in X^{(2)}$ if and only if $x^2 \in X$ and

$$\|x\|_{(2)} = \|x^2\|^{1/2}.$$

Of course $\|.\|_{(2)}$ is 2-convex. For $C > 0$, $1 \le p < \infty$, $n \in \mathbf{N}$ and $x_1, x_2, \ldots, x_n \in X$ we say that $(x_i)_{i=1}^n$ is C-equivalent to the unit vector basis of ℓ_p^n if there exist constants $A, B > 0$ with $AB \le C$ such that

$$\frac{1}{A}(\sum |a_i|^p)^{1/p} \le \| \sum a_i x_i \| \le B(\sum |a_i|^p)^{1/p}$$

for every sequence of scalars $(a_i)_{i=1}^n$. For $C > 0$, we say that X is an *asymptotic ℓ_p space* (resp. *asymptotic ℓ_p space for vectors with disjoint supports*) *with constant C* if for every n and for every sequence of vectors $(x_i)_{i=1}^n$ such that $(\mathrm{supp}\,(x_i))_{i=1}^n$ is S_1-admissible (resp. S_1-allowable), we have that $(x_i)_{i=1}^n$ are C-equivalent to the unit vector basis of ℓ_p^n.

If $(\|.\|_n)$ is a sequence of norms in c_{00} then $\Sigma(\|.\|_n)$ will denote the completion of c_{00} under the norm

$$\|x\|_{\Sigma(\|.\|_n)} = \sum_{n=1}^{\infty} \|x\|_n.$$

Fix a sequence $\alpha = (\alpha_n)_{n\in\mathbf{N}}$ of elements of $(0,1)$ with

$$0 < \inf \frac{\alpha_{n+1}}{\alpha_n} \le \sup \frac{\alpha_{n+1}}{\alpha_n} < 1 \text{ and } \sum_n \alpha_n = 1.$$

The last relationships will always be assumed whenever a sequence (α_n) will be considered in these notes. Edgington defined a sequence of norms $(\|.\|_{E,n})$ on c_{00} by

$$\|x\|_{E,0} = \|x\|_\infty, \quad \|x\|_{E,n+1}^2 = \sup\{\sum_i \|E_i x\|_{E,n}^2 : (E_i)_i \text{ is } S_1\text{-admissible}\}.$$

Then Edgington defined the norm $\|.\|_{E_\alpha}$ by

$$\|x\|_{E_\alpha} = \left(\sum_n \alpha_n \|x\|_{E,n}^2 \right)^{1/2}.$$

Let E_α denote the completion of c_{00} with respect to $\|.\|_{E_\alpha}$. It is shown in [E] that E_α is a weak Hilbert space which is not isomorphic to ℓ_2, yet it is ℓ_2-saturated. It is easy to see that the spaces constructed by Edgington are asymptotic ℓ_2 spaces for vectors with disjoint supports. The main theorem that we prove in these notes (Theorem 3.1) shows that such spaces are weak Hilbert spaces.

Let $(|.|_n)_{n\in\mathbf{N}}$ denote the sequence of the Schreier norms on c_{00}:

$$|x|_n = \sup_{S\in S_n} \sum_{j\in S} |x_j|$$

(if $x = \sum_j x_j e_j$). Then the weak Hilbert space E_α that was constructed by Edgington [E] is the 2-convexification of $\Sigma(\alpha_n|.|_n)$. One can see that $\Sigma(\alpha_n|.|_n)$ is an asymptotic ℓ_1 space for vectors with disjoint supports which is ℓ_1-saturated, yet not isomorphic to ℓ_1. In these notes we give examples of sequences of norms that can replace $(|.|_n)$ in $\Sigma(\alpha_n|.|_n)$ to obtain asymptotic ℓ_1 spaces for vectors with disjoint supports which are ℓ_1-saturated yet not isomorphic to ℓ_1. The 2-convexification of each of these spaces will give ℓ_2 saturated weak Hilbert spaces which are not isomorphic to ℓ_2.

Definition of the spaces V, W, V' and W': Let $(\theta_n)_{n\in\mathbf{N}}$ be a sequence of real numbers in $(0,1)$ with $\lim_n \theta_n = 0$ (this assumption will always be valid whenever a sequence (θ_n) will be considered in these notes) and let $s \in \mathbf{N}$. The asymptotic ℓ_1 spaces $V = T_M(\theta_n, S_n)_n$ and $W = T_{M(s)}(\theta_n, S_n)_n$ were introduced in [ADKM] (following [AD]) as the completion of c_{00} under the norms:

$$\begin{aligned}
\|x\|_V &= \|x\|_\infty \vee \sup_n \sup\{\theta_n \sum_i \|E_i x\|_V : (E_i) \text{ is } S_n \text{ allowable}\}, \\
\|x\|_W &= \|x\|_\infty \vee \sup_{n\le s} \sup\{\theta_n \sum_i \|E_i x\|_W : (E_i) \text{ is } S_n \text{ allowable}\} \\
&\qquad \vee \sup_{n\ge s+1} \sup\{\theta_n \sum_i \|E_i x\|_W : (E_i) \text{ is } S_n \text{ admissible}\},
\end{aligned}$$

respectively. These norms can also be defined as limits of appropriate sequences. For $x \in c_{00}$ let

$$\|x\|_{V,0} = \|x\|_{W,0} = \|x\|_\infty$$

and for $m \in \mathbf{N}$ define:

$$\|x\|_{V,m+1} = \|x\|_\infty \vee \sup_n \sup\{\theta_n \sum_i \|E_i x\|_{V,m} : (E_i) \text{ is } S_n \text{ allowable}\},$$

$$\|x\|_{W,m+1} = \|x\|_\infty \vee \sup_{n\le s} \sup\{\theta_n \sum_i \|E_i x\|_{W,m} : (E_i) \text{ is } S_n \text{ allowable}\}$$
$$\vee \sup_{n\ge s+1} \sup\{\theta_n \sum_i \|E_i x\|_{W,m} : (E_i) \text{ is } S_n \text{ admissible}\},$$

Then

$$\|x\|_V = \lim_m \|x\|_{V,m}, \quad \|x\|_W = \lim_m \|x\|_{W,m}.$$

Then one can construct the spaces $V' = \Sigma(\alpha_n \|.\|_{V,n})$, and $W' = \Sigma(\alpha_n \|.\|_{W,n})$. We show that V' and W' are ℓ_1-saturated asymptotic ℓ_1 spaces for vectors with disjoint supports.

It is known [CO] that if $\theta_n = \delta^n$ for some $\delta \in (0,1)$ then one can replace the "allowable" by "admissible" in the definition of $\|.\|_V$ to obtain an equivalent norm for V. For this choice of (θ_n) the variant of the norm $\|.\|_{V,m+1}$ by replacing "allowable" by "admissible" can be minorized and majorized up to a uniform multiplicative constant by the norms $\|.\|_{V,m}$ and $\|.\|_{V,m+1}$ respectively. This is enough to conclude that the new norms lead to an equivalent norm for V' (see [B]).

3. The main Theorems

The following results is the main tool of our paper for constructing weak Hilbert spaces. It is based on results of Johnson, and it was proved (but not stated) in [NTJ], Section 4. Since it is hard for a non-specialist to extract it from the existing literature, we outline it here.

Theorem 3.1. *If X is an asymptotic ℓ_2 space for vectors with disjoint supports then X is a weak Hilbert space.*

Proof Recall that the **fast growing hierarchy** is a sequence of functions on the natural numbers (g_n) which is defined inductively by: $g_0(n) = n+1$, and for $i \ge 0$, $g_{i+1}(n) = g_i^n(n)$ where $g^n = g \circ g \circ \cdots \circ g$ is the n-fold iteration of g and $g^0 = I$.

Step I

If X is asymptotic-ℓ_2 with constant C for vectors with disjoint supports, then for every $i \geq 0$ any $g_i(n)$ normalized disjointly supported vectors with supports after n are C^i-equivalent to the unit vector basis of ℓ_2.

We proceed by induction on i with the case $i = 0$ being trivial. So, assume Step I holds for some $i \geq 0$ and let $\{x_k : 1 \leq k \leq g_{i+1}(n)\}$ be a sequence of disjointly supported vectors in X with supports after n. For $1 \leq j \leq n$ let

$$E_j = \{k : g_i^{j-1}(n) \leq k \leq g_i^j - 1\}.$$

Then,

$$\| \sum_{k=1}^{g_{i+1}^n(n)} x_k \| \overset{C}{\approx} \left(\sum_{j=1}^{n} \| \sum_{k \in E_j} x_k \|^2 \right)^{1/2}$$

Applying the induction hypotheses to each sum on the right we continue this equivalence as

$$\overset{C^{i+1}}{\approx} \left(\sum_{k=1}^{g_{i+1}^n(n)} \|x_k\|^2 \right)^{1/2} .$$

Step II

If X is asymptotic-ℓ_2 with constant C for vectors with disjoint supports then every n-dimensional subspace of X supported after n is $8C^3$-isomorphic to a Hilbert space and $8C^3$-complemented in X.

If E is a $5^{(5^n)}$-dimensional subspace of X supported after n, then by a result of Johnson (See Proposition V.6 of [CS]) there is a subspace G of X spanned by $\leq g_3(n)$ disjointly supported vectors supported after n and an operator $V : E \to G$ with $\|Vx - x\| \leq \frac{1}{2}\|x\|$, for all $x \in E$. Now, by Step I, we have that E is $2C^3$-isomorphic to a Hilbert space. It follows [J2] that every 5^n-dimensional space of X^* supported after n is $4C^3$-isomorphic to a Hilbert space and $4C^3$-complemented in X^*. Therefore, every n-dimensional subspace of X supported after n is $8C^3$-isomorphic to a Hilbert space and $8C^3$-complemented in X.

Step III

Every asymptotic-ℓ_2 space for vectors with disjoint supports is a weak Hilbert space.

If E is a $2n$-dimensional subspace of X, let $F =: E \cap (\text{span}_{k\geq n} e_k)$. Then F is supported after n and $\dim F \geq n$ implies F is K-isomorphic to a Hilbert space and K-complemented in X by Step II, where $K = 8C^3$. It follows from Definition 2.1 that X is a weak Hilbert space. □

The 2-convexification of certain Tsirelson spaces for obtaining weak Hilbert spaces has been used before (see [P], [ADKM]). More generally we have the following:

Corollary 3.2. *If X is an asymptotic ℓ_1 space for vectors with disjoint supports then $X^{(2)}$ is a weak Hilbert space.*

Proof: Let $(X, \|.\|)$ be an asymptotic ℓ_1 space for vectors with disjoint supports. Then there exists $C > 0$ such that for every sequence of vectors (x_i) with $(\text{supp}\, x_i)$ being S_1-allowable, we have that $C \sum \|x_i\| \leq \|\sum x_i\|$. It suffices to prove that $X^{(2)}$ is an asymptotic ℓ_2 space for vectors with disjoint support. Let (y_i) be a sequence of vectors in $X^{(2)}$ with $(\text{supp}\, y_i)$ being S_1-allowable. Then

$$C^{1/2}(\sum \|y_i\|_{(2)}^2)^{1/2} = C^{1/2}(\sum \|y_i^2\|)^{1/2} \leq \|\sum y_i^2\|^{1/2} = \|(\sum y_i^2)^{1/2}\|.$$

Also,

$$\|(\sum y_i^2)^{1/2}\|_{(2)} = \|\sum y_i^2\|^{1/2} \leq (\sum \|y_i^2\|)^{1/2} = (\sum \|y_i\|_{(2)}^2)^{1/2}.$$

□

The spaces V, W, V' and W' are asymptotic ℓ_1 spaces for vectors with disjoint supports. Indeed, this is obvious for V and W. To see this for V' let $n \in \mathbf{N}$ and vectors $(x_i)_{i=1}^n$ with disjoint supports with $n \leq x_i$ for all i. Then:

$$\begin{aligned} \|\sum_{i=1}^n x_i\| &= \sum_{m=1}^\infty \alpha_m \|\sum_{i=1}^n x_i\|_{V,m} \geq \sum_{m=1}^\infty \alpha_m \theta_1 \sum_{i=1}^n \|x_i\|_{V,m-1} \\ &\geq \theta_1 u \sum_{i=1}^n \sum_{m=1}^\infty \alpha_{m-1} \|x_i\|_{V,m-1} \geq \theta_1 u \sum_{i=1}^n \|x_i\|_{V'} \end{aligned}$$

The proof for W' is similar. Thus $V^{(2)}$, $W^{(2)}$, $V'^{(2)}$ and $W'^{(2)}$ are weak Hilbert spaces.

Proposition 3.3. *The spaces V and W do not contain an isomorph of ℓ_1. The spaces V' and W' are ℓ_1-saturated without being isomorphic to ℓ_1.*

The following Lemma will be used in the proof of Proposition 3.3.

Lemma 3.4. *For every $m \in \mathbf{N}$ the completion of $(c_{00}, \|.\|_{V,m})$ is a c_0-saturated space.*

In order to prove this Lemma we need a result of Fonf along with the notion of the boundary.

Definition 3.5. *A subset B of the unit sphere of the dual of a Banach space X is called a boundary for X if for every $x \in X$ there exists $f \in B$ such that $f(x) = \|x\|$.*

Theorem 3.6. *(*[F1]*, see also* [F2]*,* [H] [DGZ]*) Every Banach space with a countable boundary is c_0-saturated.*

Proof of Lemma 3.4 Define inductively on $i \leq m$ the sets K^i of the unit ball of the dual of $(c_{00}, \|.\|_{V,m})$. Let $K^0 = \{\pm e_n : n \in \mathbf{N}\}$. For $i < m$ if K^i has been defined then let

$$K^{i+1} = K^i \cup \{\theta_k(f_1 + \cdots + f_r) : F_j \in K^m, \text{ for all } j, (\text{supp } f_j)_{j=1}^r \text{ is } S_k \text{ allowable } k = 1, 2, \dots\}.$$

Then K^m is a norming set for $(c_{00}, \|.\|_{V,m})$:

$$\|x\|_{V,m} = \sup\{|f(x)| : f \in K^m\}.$$

It is easy to see that $K^m \cup \{0\}$ is a closed set in the topology of pointwise convergence, since each S_k is closed in the same topology and $\lim_k \theta_k = 0$. The previous Theorem of Fonf finishes the proof of the Lemma. □

For Tsirelson's space a similar result to Lemma 3.4 was proved in [P] (Lemma 13.8); the proof was by a simple direct calculation without the use of Fonf's result.

Proof of Proposition 3.3 The statement for V and W is proved in [ADKM].

V' is ℓ_1 saturated: Let (x_i) be an arbitrary block basis of V'. It is enough to construct a

normalized (in V') block basis (v_i) of (x_i) and an increasing sequence of positive integers $1 = p_1 < p_2 < p_3 < \cdots$ such that

$$\sum_{m=p_i}^{p_{i+1}-1} \alpha_m \|v_i\|_{V,m} \geq \frac{1}{2}$$

for all i. Once this is done then for $(\lambda_i) \in c_{00}$

$$\begin{aligned} \|\sum_{i=1}^n \lambda_i v_i\|_{V'} &= \sum_{m=1}^\infty \alpha_m \|\sum_{i=1}^n \lambda_i v_i\|_{V,m} = \sum_{j=1}^\infty \sum_{m=p_j}^{p_{j+1}-1} \alpha_m \|\sum_{i=1}^n \lambda_i v_i\|_{V,m} \\ &\geq \sum_{j=1}^\infty \sum_{m=p_j}^{p_{j+1}-1} \alpha_m \|\lambda_j v_j\|_{V,m} \geq \frac{1}{2} \sum_{j=1}^\infty |\lambda_j| \end{aligned}$$

which shows that (v_i) is equivalent to the unit vector basis of ℓ_1. In order to choose such (v_i) and (p_i) we use that for every $m \in \mathbf{N}$ the norms $\|.\|_{V,m}$ and $\|.\|_{V,m+1}$ are not equivalent. Thus for every $m, M, K \in \mathbf{N}$ there is u in the span of (x_i) with

$$M \leq u, \quad \|u\|_{V,m} < \frac{1}{4}, \text{ and } \|u\|_{V,m+1} \geq K.$$

Then

$$\sum_{i=1}^m \alpha_i \|u\|_{V,i} < \frac{1}{4} \text{ and } \sum_{i=1}^\infty \alpha_i \|u\|_{V,i} \geq \alpha_{m+1} K.$$

Let $v = \frac{u}{\|u\|_{V'}}$. By taking K large enough we can assume that

$$\sum_{i=1}^m \alpha_i \|v\|_{V,i} < \frac{1}{4}.$$

Also choose $m' > m$ with

$$\sum_{i=m'+1}^\infty \alpha_i \|v\|_{V,i} < \frac{1}{4}.$$

Thus

$$\sum_{i=m+1}^{m'} \alpha_i \|v\|_{V,i} \geq \frac{1}{2}.$$

It only remains to show that for every $m \in \mathbf{N}$ the norms $\|.\|_{V,m}$ and $\|.\|_{V,m+1}$ are not equivalent on the span of (x_i). By the previous Lemma there is a block sequence (y_i) of (x_i)

such that $\|y_i\|_{V,m} = 1$ for all i and $((y_i), \|.\|_{V,m})$ is 2-equivalent to the unit vector basis of c_0. For $n \in \mathbf{N}$ let $k \in \mathbf{N}$ with $n \leq y_{k+1} < y_{k+2} < \cdots < y_{k+n}$. Thus

$$\| \sum_{i=k+1}^{k+n} y_i \|_{V,m} \leq 2$$

yet

$$\| \sum_{i=k+1}^{k+n} y_i \|_{V,m+1} \geq \theta_1 n.$$

This proves the result.

W' is ℓ_1-saturated: Similar.

V' is not isomorphic to ℓ_1: If the statement were false then the basis of V' would be isomorphic to the unit vector basis of ℓ_1 (since every normalized unconditional basic sequence in ℓ_1 is equivalent to the usual unit basis of ℓ_1, [LP]). Observe that for $x \in c_{00}$,

$$\|x\| = \sum_m \alpha_m \|x\|_{V,m} \leq \sup_m \|x\|_{V,m} = \|x\|_V.$$

By [ADKM] the norm of V can become arbitrarily smaller that the ℓ_1 norm on certain vectors.

W' is not isomorphic to ℓ_1: Similar. □

Remark 3.7. *Note that a space X with an unconditional basis contains ℓ_1 if and only if $X^{(2)}$ contains ℓ_2. Since V and W do not contain an isomorph of ℓ_1, we obtain that $V^{(2)}$ and $W^{(2)}$ are weak Hilbert spaces which do not contain an isomorph of ℓ_2. Since the spaces V' and W' are ℓ_1 saturated without being isomorphic to ℓ_1, we obtain that the spaces ${V'}^{(2)}$, and ${W'}^{(2)}$ are ℓ_2-saturated weak Hilbert spaces which are not isomorphic to ℓ_2.*

Thus the essential properties of the space E_α constructed by Edgington are shared by ${V'}^{(2)}$ and ${W'}^{(2)}$.

Theorem 3.8. *Let $(\theta_n) \subset (0,1)$ with $\lim_n \theta_n^{1/n} = 1$, $s \in \mathbf{N}$, and $\beta = (\beta_n) \subset (0,1)$ with $\sum_n \beta_n = 1$ and $0 < \inf \frac{\beta_{n+1}}{\beta_n} \leq \sup \frac{\beta_{n+1}}{\beta_n} < 1$. Then ${V'}^{(2)}$ and ${W'}^{(2)}$ are not isomorphic to E_β.*

Proof: Let $T : X \to E_\beta$ be an isomorphism where X is either $V'^{(2)}$ or $W'^{(2)}$. Since T is an isomorphism there exists $C > 0$ such that

$$\frac{1}{C}\|Tx\|_{E_\beta} \leq \|x\|_X \leq C\|Tx\|_{E_\beta}$$

for all $x \in c_{00}$. Also, by [E] (proof of Theorem 7) there exists $\delta > 0$ such that

$$\|Tx\|_{E_\beta} \leq C\|Tx\|_{T^{(2)}(\delta,S_1)}.$$

Thus for $x \in c_{00}$

$$\|x\|_X \leq C^2\|Tx\|_{T^{(2)}(\delta,S_1)}. \tag{1}$$

Since the unit vector basis (e_i) of X is weakly null, we can select a subsequence (e_{k_i}) of (e_i), a block sequence (u_i) in $T^{(2)}(\delta, S_1)$ and a number $K > 0$ such that:

$$\|T(e_{k_i}) - u_i\|_{T^{(2)}(\delta,S_1)} < \frac{\varepsilon}{2^i} \text{ and } \frac{1}{K} \leq \|u_i\|_{T^{(2)}(\delta,S_1)} \leq K \text{ for all } i,$$

where $\varepsilon > 0$ will be chosen later. Let $n \in \mathbf{N}$ to be selected later. Let $(x_i)_{i\in I} \subset (0, 1)$ for some $I \in S_n$ so that

$$\sum_{i\in I} x_i^2 = 1, \text{ and } \|\sum_{i\in I} x_i^2 \frac{u_i^2}{\|u_i^2\|_{T(\delta,S_1)}}\|_{T(\delta,S_1)} \leq \delta^n + \varepsilon$$

([OTW] Theorem 5.2 (a)). Then

$$\begin{aligned}
\|T(\sum_{i\in I} x_i e_{k_i})\|_{T^{(2)}(\delta,S_1)} &\leq \|\sum_{i\in I} x_i u_i\|_{T^{(2)}(\delta,S_1)} + \sum_{i\in I} x_i\|Te_{k_i} - u_i\|_{T^{(2)}(\delta,S_1)} \\
&\leq \|\sum_{i\in I} x_i^2 u_i^2\|^{1/2}_{T(\delta,S_1)} + \varepsilon \\
&\leq K\|\sum_{i\in I} x_i^2 \frac{u_i^2}{\|u_i^2\|_{T(\delta,S_1)}}\|^{1/2}_{T(\delta,S_1)} + \varepsilon \\
&\leq K(\delta^n + \varepsilon)^{1/2} + \varepsilon.
\end{aligned}$$

On the other hand if $Y = V'$ when $X = V'^{(2)}$ or $Y = W'$ when $X = W'^{(2)}$ then

$$\|\sum_{i\in I} x_i e_{k_i}\|_X = \|\sum_{i\in I} x_i^2 e_{k_i}\|^{1/2}_Y \geq (\sum_{m=1}^{\infty} \beta_m\theta_n \sum_{i\in I} x_i^2)^{1/2} = \sqrt{\theta_n}$$

Therefore (1) gives

$$\sqrt{\theta_n} \leq C^2(K(\delta^n + \varepsilon)^{1/2} + \varepsilon).$$

But since $\lim_n \theta_n^{1/n} = 1$, n and ε can be chosen so that this inequality fails. □

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Remark After submitting the above paper, the authors noticed that the proof of Proposition 3.3 yields the following more general result:

Proposition *Let a sequence of norms $(\|.\|_m)_{m\in\mathbf{N}}$ on c_{00}, a sequence of positive numbers $(\alpha_m)_{m\in\mathbf{N}}$ and a positive number δ such that the following conditions are satisfied:*

1. *For every $m \in \mathbf{N}$, the unit vector basis of $(c_{00}, \|.\|_m)$ is unconditional.*
2. *For every $m, n \in \mathbf{N}$ and for every vectors $(x_i)_{i=1}^n$ in c_{00} with $n \le x_1 < x_2 < \cdots < x_n$ we have that*

$$\|\sum_{i=1}^n x_i\|_{m+1} \ge \delta \sum_{i=1}^n \|x_i\|_m.$$

3. *For every $m \in \mathbf{N}$, c_0 is finitely block represented in every block subspace of $(c_{00}, \|.\|_m)$.*
4. *For every $x \in c_{00}$, $\sup_m \|x\|_m \le \|x\|_{\ell_1}$ and the norm $\sup_m \|.\|_m$ is not equivalent to the ℓ_1 norm $\|.\|_{\ell_1}$.*
5. $0 < \inf_m \frac{\alpha_{m+1}}{\alpha_m} \le \sup_m \frac{\alpha_{m+1}}{\alpha_m} < 1$.

Define a norm $\|.\|$ *on* c_{00} *by* $\|x\| = \sum_m \alpha_m \|x\|_m$. *Then* $(c_{00}, \|.\|^{(2)})$ *is an weak Hilbert space, which is* ℓ_2*-saturated, yet not isomorphic to* ℓ_2.

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