Distorting Mixed Tsirelson Spaces*

G. Androulakis † E. Odell ‡

Abstract: Any regular mixed Tsirelson space $T(\theta_n, S_n)_{\mathbf{N}}$ for which $\frac{\theta_n}{\theta^n} \to 0$, where $\theta = \lim_n \theta_n^{1/n}$, is shown to be arbitrarily distortable. Certain asymptotic ℓ_1 constants for those and other mixed Tsirelson spaces are calculated. Also a combinatorial result on the Schreier families $(S_\alpha)_{\alpha<\omega_1}$ is proved and an application is given to show that for every Banach space X with a basis (e_i), the two Δ-spectrums $\Delta(X)$ and $\Delta(X, (e_i))$ coincide.

1 Introduction

A Banach space X with basis (e_i) is asymptotic ℓ_1 if there exists $\delta > 0$ such that for all n and block bases $(x_i)_1^n$ of $(e_i)_n^\infty$,

$$\| \sum_{i=1}^n x_i \| \geq \delta \sum_{i=1}^n \|x_i\|. \tag{1}$$

Such a space need not contain ℓ_1 as witnessed by Tsirelson's famous space T. The complexity of the asymptotic ℓ_1 structure within X can be measured by certain constants $\delta_\alpha(e_i)$ for $\alpha < \omega_1$. $\delta_1(e_i)$ is the largest $\delta > 0$ satisfying (1) above. Subsequent δ_α's are defined by a similar formula where $(x_i)_1^n$ ranges over "α-admissible" block bases (all terms are precisely defined in section 2). These notions were developed in [OTW] where, in addition, $\delta_\alpha(y_i)$ was considered, for a block basis (y_i) of (e_i). In this setting, (y_i) becomes the reference frame and one naturally has $\delta_\alpha(y_i) \geq \delta_\alpha(e_i)$. These constants can perhaps increase by passing to further block bases and this leads to the notion of the Δ-spectrum of X, $\Delta(X)$. Roughly, $\Delta(X)$ is the set of all $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ where γ_α is the stabilization of $\delta_\alpha(y_i)$ for (y_i) some block basis of (e_i). Alternatively by keeping (e_i) as the reference frame, in a similar manner we obtain $\Delta(X, (e_i))$. In section 3 we prove that these two notions coincide, $\Delta(X) = \Delta(X, (e_i))$.

Argyros and Deliyanni [AD] constructed the first example of an asymptotic ℓ_1 arbitrarily distortable Banach space by constructing "mixed Tsirelson spaces" and proving that such spaces can be arbitrarily distortable. In section 4 we consider the simplest class of mixed Tsirelson spaces $X = T(\theta_n, S_n)_{n\in\mathbf{N}}$ where $\theta_n \to 0$ and $\sup_n \theta_n < 1$. These are reflexive asymptotic ℓ_1 spaces having a 1-unconditional basis (e_i). Also we may assume $\theta \equiv \theta_n^{1/n}$ exists. We prove that if $\frac{\theta_n}{\theta^n} \to 0$ then X is arbitrarily distortable. In particular, this happens if $\theta = 1$. Thus, for example, $T(\frac{1}{n+1}, S_n)_{\mathbf{N}}$ is an arbitrarily distortable space. We also calculate the asymptotic constants $\ddot{\delta}_\alpha(X)$ for these spaces along with the spectral index $I_\Delta(X)$. $\ddot{\delta}_\alpha(X)$ is the supremum of $\delta_\alpha((x_i), |\cdot|)$ under all equivalent norms on X and $I_\Delta(X)$ is the first ordinal α for which $\ddot{\delta}_\alpha(X) < 1$.

*$Keywords$: Schreier families, Δ-spectrum, mixed Tsirelson norms, arbitrarily distortable Banach space.

†Part of this paper also appears in the first author's Ph.D. thesis which is being prepared under the supervision of Prof. H. Rosenthal at the University of Texas at Austin.

‡Research supported by NSF and TARP

2 Preliminaries

$X, Y, Z, \ldots$ shall denote separable infinite dimensional Banach spaces. *All the spaces we consider will have bases.* Every Banach space with a basis can be viewed as the completion of c_{00} (the linear space of finitely supported real valued sequences) under a certain norm. (e_i) will denote the unit vector basis for c_{00} and whenever a Banach space $(X, \|\cdot\|)$ with a basis is regarded as the completion of $(c_{00}, \|\cdot\|)$, (e_i) will denote this (normalized) basis. If $x \in c_{00}$ and $E \subseteq \mathbb{N}$, $Ex \in c_{00}$ is the restriction of x to E; $Ex(j) = x(j)$ if $j \in E$ and 0 otherwise. Also the *support* of x, $\mathrm{supp}\,(x)$, (w.r.t. (e_i)) is the set $\{j \in \mathbb{N} : x(j) \neq 0\}$. The *range* of x, $\mathrm{ran}\,(x)$, (w.r.t. (e_i)) is the smallest interval which contains $\mathrm{supp}\,(x)$. If x, x_1, $x_2, \ldots$ are vectors in X (and $k \in \mathbb{N}$) then we say that x is an average of $(x_i)_i$ (of length k) if there exists $F \subset \mathbb{N}$ with $x = \frac{1}{|F|}\sum_{i \in F} x_i$ (and $\mid F \mid= k$). We say that a sequence (y_i) is a *(convex) block sequence of* (x_i) if for all i, $y_i = \sum_{j=m_i}^{m_{i+1}-1} \alpha_j x_j$ for some sequence $m_1 < m_2 < \ldots$, of integers and $(\alpha_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ (resp. with $\alpha_j \geq 0$ for all j, and $\sum_{j=m_i}^{m_{i+1}-1} \alpha_j = 1$ for all i). If (x_i) is a block basis of (y_i) we write $(x_i) \prec (y_i)$. $X \prec Y$ shall mean that X has a basis which is a block basis of a certain basis for Y, when the given bases are understood. For $\lambda > 1$, $(X, \|\cdot\|)$ is λ*-distortable* if there exists an equivalent norm $|\cdot|$ on X so that for all $Y \prec X$

$$D(Y, |\cdot|) \equiv \sup\{\frac{|\, y\,|}{|\, z\,|} : y, z \in Y, \|y\| = \|z\| = 1\} \geq \lambda.$$

X is *distortable* if it is λ-distortable for some $\lambda > 1$ and *arbitrarily distortable* if it is λ-distortable for all $\lambda > 1$. X is of *D-bounded distortion* if for all equivalent norms $|\cdot|$ on X and for all $Z \prec X$ there exists $Y \prec Z$ with $D(Y, |\cdot|) \leq D$. Note that if $C = \inf\{\frac{\|y\|}{|y|} : y \in Y, y \neq 0\}$ then

$$C \mid y \mid \leq \|y\| \leq D(Y, |\cdot|)C \mid y \mid \text{ for all } y \in Y. \tag{2}$$

For more information on distortion we recommend the reader consult the following papers: [S], [MT], [OS1], [OS2], [OS3], [Ma], [T], [OTW].

Asymptotic ℓ_1 Banach spaces are defined by (1) in section 1 (for another approach to asymptotic structure see [MMT]). These spaces were studied in [OTW] where certain asymptotic constants were introduced. We shall recall the relevant definitions but first we need to recall the definition of the Schreier sets S_α, $\alpha < \omega_1$ [AA]. For $F, G \subset \mathbb{N}$, we write $F < G$ when $\max(F) < \min(G)$ or one of them is empty, and we write $n \leq F$ instead of $\{n\} \leq F$. Also for $x, y \in c_{00}$, $x < y$ means $\mathrm{ran}\,(x) < \mathrm{ran}\,(y)$.

Definition 2.1 $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. *If* $\alpha < \omega_1$ *and* S_α *has been defined,*
$S_{\alpha+1} = \{\cup_1^n F_i : n \in \mathbb{N}, n \leq F_1 < F_2 < \cdots < F_n$ *and* $F_i \in S_\alpha$ *for* $1 \leq i \leq n\}$.
If α *is a limit ordinal choose* $\alpha_n \nearrow \alpha$ *and set* $S_\alpha = \{F : n \leq F \in S_{\alpha_n}$ *for some* $n\}$.

If $(E_i)_1^\ell$ is a finite sequence of non-empty subsets of $\mathbb{N}$ and $\alpha < \omega_1$ then we say that $(E_i)_1^\ell$ *is* α*-admissible* if $E_1 < \cdots < E_\ell$ and $(\min\ E_i)_1^\ell \in S_\alpha$. If (e_i) is a basic sequence and $(x_i)_1^\ell \prec (e_i)$ then $(x_i)_1^\ell$ is α*-admissible with respect to* (e_i) if $(\mathrm{ran}\,(x_i))_1^\ell$ is α-admissible where the range of x, $\mathrm{ran}\,(x)$, is w.r.t (e_i). If $x \in \mathrm{span}\,(x_i)$, then x is α*-admissible w.r.t.* (x_i) if $\mathrm{supp}\,(x)$ (w.r.t. (x_i)) $\in S_\alpha$. Also if $x \in \mathrm{span}\,(x_i)$ then x *is a 1-admissible average of* (x_i) *w.r.t.* (e_i) if there exists a finite set $F \subset \mathbb{N}$ such that $x = \frac{1}{|F|}\sum_{i \in F} x_i$ and $(x_i)_{i \in F}$ is 1-admissible w.r.t. (e_i). Note that if x is a

1-admissible average of (x_i) w.r.t. (e_i) and for some $\alpha < \omega_1$ each x_i is α-admissible w.r.t. (e_i) then x is $\alpha+1$-admissible w.r.t. (e_i). Thus if (x_i) is a basis for X then X is asymptotic ℓ_1 iff

$$0 < \delta_1(x_i) \equiv \delta_1(X) \equiv \delta_1(X, \|\cdot\|) = \sup \{\ \delta \geq 0 : \|\sum_1^n y_i\| \geq \delta \sum_1^n \|y_i\| \text{ whenever } (y_i)_1^n \prec (x_i) \text{ and } (y_i)_1^n \text{ is 1-admissible w.r.t. } (x_i)\}.$$

In [OTW] this definition was extended as follows: For $\alpha < \omega_1$

$$\delta_\alpha(x_i) \equiv \delta_\alpha(X) \equiv \delta_\alpha(X, \|\cdot\|) = \sup \{\ \delta \geq 0 : \|\sum_1^n y_i\| \geq \delta \sum_1^n \|y_i\| \text{ whenever } (y_i)_1^n \prec (x_i) \text{ and } (y_i)_1^n \text{ is } \alpha\text{-admissible w.r.t. } (x_i)\}.$$

Observation 2.2 *Note that if we have two equivalent norms $\|\cdot\|$, $|\!|\!|\cdot|\!|\!|$ on X and for some $c, C > 0$, $c|\!|\!|x|\!|\!| \leq \|x\| \leq C|\!|\!|x|\!|\!|$ for all $x \in X$, then for all $\alpha < \omega_1$,*

$$\frac{c}{C}\delta_\alpha(X, |\!|\!|\cdot|\!|\!|) \leq \delta_\alpha(X, \|\cdot\|) \leq \frac{C}{c}\delta_\alpha(X, |\!|\!|\cdot|\!|\!|).$$

In problems of distortion one is concerned with block bases and equivalent norms. Thus we also consider [OTW]

$$\begin{aligned} \dot{\delta}_\alpha(x_i) &= \sup\{\delta_\alpha(y_i) : (y_i) \prec (x_i)\} \text{ and} \\ \ddot{\delta}_\alpha(x_i) &= \sup\{\dot{\delta}_\alpha((x_i), |\cdot|) : |\cdot| \text{ is an equivalent norm on } X\}. \end{aligned}$$

If $(y_i) \prec (x_i)$ then $\delta_\alpha(y_i) \geq \delta_\alpha(x_i)$. This is because each S_α is *spreading* (if $(n_i)_1^k \in S_\alpha$ and $m_1 < \cdots < m_k$ with $n_i \leq m_i$ for all $i = 1, \ldots, k$, then $(m_i)_1^k \in S_\alpha$). This leads to the following definition [OTW].

Definition 2.3 *A basic sequence (y_i) Δ-stabilizes $\gamma = (\gamma_\alpha)_{\alpha<\omega_1} \subseteq \mathbb{R}$ if there exists $\varepsilon_n \searrow 0$ so that for all $\alpha < \omega_1$ there exists $m \in \mathbb{N}$ so that for all $n \geq m$ if $(z_i) \prec (y_i)_n^\infty$ then $|\ \delta_\alpha(z_i) - \gamma_\alpha\ | < \varepsilon_n$.*

Remark It is automatic from the definition that if (y_i) Δ-stabilizes γ then for all $\alpha < \omega_1$, $\gamma_\alpha = \sup\{\delta_\alpha(z_i) : (z_i) \prec (y_i)\}$. Furthermore if $(z_i) \prec (y_i)$ then (z_i) Δ-stabilizes γ.

It is shown in [OTW] that if X has a basis (x_i) and $(y_i) \prec (x_i)$ then there exists $(z_i) \prec (y_i)$ and $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ so that (z_i) Δ-stabilizes γ.

Definition 2.4 *Let X have a basis (x_i). The Δ-spectrum of X, $\Delta(X)$, is defined to be the set of all γ's so that (y_i) stabilizes γ for some $(y_i) \prec (x_i)$. We also define $\ddot{\Delta}(X) = \cup\{\Delta(X, |\cdot|) : |\cdot|$ is an equivalent norm on $X\}$.*

We have that $\Delta(X) \neq \emptyset$ and it is easy to see that $\ddot{\delta}_\alpha(X) = \sup\{\gamma_\alpha : \gamma \in \ddot{\Delta}(X)\}$

Theorem 2.5 *[OTW] Let X have a basis (x_i).*

1. *If $\gamma \in \Delta(X)$ then γ_α is a continuous decreasing function of α. Also $\gamma_{\alpha+\beta} \geq \gamma_\alpha\gamma_\beta$ for all $\alpha, \beta < \omega_1$.*

2. *For all $\alpha < \omega_1$ and $n \in \mathbb{N}$, $\ddot{\delta}_{\alpha\cdot n}(X) = (\ddot{\delta}_\alpha(X))^n$.*

3. X does not contain ℓ_1 iff $\ddot{\delta}_\alpha(X) = 0$ for some $\alpha < \omega_1$.

Definition 2.6 *Let X have a basis (x_i). The* spectral index *$I_\Delta(X)$ is defined by $I_\Delta(X) = \inf\{\alpha < \omega_1 : \ddot{\delta}_\alpha(X) < 1\}$ if such an α exists and $I_\Delta(X) = \omega_1$, otherwise.*

Definition 2.7 Mixed Tsirelson Norms *[AD] Let $F \subseteq \mathbb{N}$. Let $(\alpha_n)_{n\in F}$ be a set of countable ordinals and $(\theta_n)_{n\in F} \subset (0,1)$. The mixed Tsirelson space $T(\theta_n, S_{\alpha_n})_{n\in F}$ is the completion of c_{00} under the implicit norm*

$$\|x\| = \|x\|_\infty \vee \sup_{q\in\mathbb{N}} \sup\{\theta_q \sum_1^n \|E_i x\| : (E_i)_1^n \text{ is an } \alpha_q\text{-admissible sequence of sets }\}.$$

It is proved in [AD] that such a norm exists. They also proved that $T(\theta_n, S_{a_n})_{n\in F}$ is reflexive if F is finite or $\lim_{F\ni n\to\infty} \theta_n = 0$. (e_n) is a 1-unconditional basis for $T(\theta_n, S_{\alpha_n})$ so we can restrict the E_i's in the above definition to be *intervals*. It is worth noting that T, Tsirelson's space [Ts] as described in [FJ] satisfies $T = T(1/2, S_1) = T(1/2^n, S_n)_{\mathbb{N}}$.

3 A property of the Δ-spectrum

The definition of $\delta_\alpha(x_i)$ is w.r.t. the coordinate system (x_i). In [OTW] the following notion is also introduced:

Definition 3.1 *Let (e_i) be a basis for X and let $(x_i) \prec (e_i)$. For $\alpha < \omega_1$ we define*

$$\delta_\alpha((x_i),(e_i)) = \sup \{ \delta \ge 0 : \|\sum_1^n y_i\| \ge \delta \sum_1^n \|y_i\| \text{ whenever } (y_i)_1^n \prec (x_i) \text{ and } (y_i)_1^n \text{ is } \alpha\text{-admissible w.r.t. } (e_i)\}.$$

If $(y_i) \prec (e_i)$ we say that (y_i) $\Delta_{(e_i)}$-stabilizes $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ if there exists $\varepsilon_n \searrow 0$ so that for all $\alpha < \omega_1$ there exists $m \in \mathbb{N}$ so that if $n \ge m$ and $(z_i) \prec (y_i)_n^\infty$ then $|\ \delta_\alpha((z_i),(e_i)) - \gamma_\alpha\ | < \varepsilon_n$. Let $\Delta(X,(e_i))$ be the set of all γ's so that (y_i) $\Delta_{(e_i)}$-stabilizes γ for some $(y_i) \prec (e_i)$.

One can show, by the same arguments used to establish the analogous result for $\Delta(X)$ [OTW], that for all $(x_i) \prec (e_i)$ there exists $(y_i) \prec (x_i)$ and $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ so that (y_i) $\Delta_{(e_i)}$-stabilizes γ. In particular, $\Delta(X,(e_i))$ is non-empty.

In this section we prove that the Δ-stabilization and the $\Delta_{(e_i)}$-stabilization are actually the same notions. More precisely we prove

Theorem 3.2 *Let X have a basis (e_i) and let $(x_i) \prec (e_i)$ so that (x_i) $\Delta_{(e_i)}$-stabilizes $\bar{\gamma} \in \Delta(X,(e_i))$ and (x_i) Δ-stabilizes $\gamma \in \Delta(X)$. Then $\bar{\gamma} = \gamma$. Hence $\Delta(X) = \Delta(X,(e_i))$.*

First we need a combinatorial result. $[\mathbb{N}]$ denotes the set of infinite subsequences of $\mathbb{N}$. If $N = (n_i) \in [\mathbb{N}]$ then $S_\alpha(N) = \{(n_i)_{i\in F} : F \in S_\alpha\}$ and $[N]$ is the set of infinite subsequences of N.

Proposition 3.3 *Let $N \in [\mathbb{N}]$. Then there exists $L = (\ell_i) \in [N]$ so that for all $\alpha < \omega_1$,*

$$(\ell_i)_{i\in F} \in S_\alpha \Rightarrow (\ell_{i+1})_{i\in F} \in S_\alpha(N).$$

Proof Let $N = (n_i)$. We shall choose $M = (m_i) \in [N]$ and then prove by induction on α that $L = (\ell_i)$ satisfies the proposition where $\ell_i = n_{m_i}$. Let $m_1 = n_1$. If m_k has been defined set $m_{k+1} = n_{m_k}$.

The case $\alpha = 0$ is trivial.

Assume the result holds for α and that $(n_{m_i})_{i \in F} \in S_{\alpha+1}$. Thus there exists $k \in \mathbb{N}$ and $n_{m_k} \leq E_1 < E_2 < \cdots E_{n_{m_k}}$ (some possibly empty) so that $E_j \in S_\alpha$ for all j and $(n_{m_i})_{i \in F} = \cup_1^{n_{m_k}} E_j$. For each j let $E_j = (n_{m_i})_{i \in F_j}$. Then $n_{n_{m_k}} = n_{m_{k+1}} \leq (n_{m_{i+1}})_{i \in F} = \cup_1^{n_{m_k}} (n_{m_{i+1}})_{i \in F_j}$ and for all j, $(n_{m_{i+1}})_{i \in F_j} \in S_\alpha(N)$. Therefore $(n_{m_{i+1}})_{i \in F} \in S_{\alpha+1}(N)$.

If α is a limit ordinal and $\alpha_n \nearrow \alpha$ are the ordinals used to define S_α and the result holds for all $\beta < \alpha$ (so in particular for each α_n), let $(n_{m_i})_{i \in F} \in S_\alpha$. Thus for some $k \in \mathbb{N}$, $k \leq \min(n_{m_i})_{i \in F} \equiv n_{m_{i_0}} \leq (n_{m_i})_{i \in F} \in S_{\alpha_k}$. Hence $n_k \leq n_{n_{m_{i_0}}} = n_{m_{i_0+1}} \leq (n_{m_{i+1}})_{i \in F} \in S_{\alpha_k}(N)$ therefore $(n_{m_{i+1}})_{i \in F} \in S_\alpha(N)$. □

As a corollary we obtain a result of independent interest.

Corollary 3.4 *Let $N \in [\mathbb{N}]$. Then there exists $L = (\ell_i) \in [N]$ so that for all $\alpha < \omega_1$,*

$$(\ell_i)_{i \in F} \in S_\alpha \Rightarrow (\ell_i)_{i \in F \setminus \min(F)} \in S_\alpha(N).$$

Proof Let L be as in proposition 3.3. Let $F = (f_1 < f_2 < \cdots < f_r)$ with $(\ell_i)_{i \in F} \in S_\alpha$. Thus $(\ell_{f_1+1}, \ell_{f_2+1}, \ldots, \ell_{f_r+1}) \in S_\alpha(N)$. Since $f_1 + 1 \leq f_2$, $f_2 + 1 \leq f_3, \ldots$ and $S_\alpha(N)$ is both spreading and hereditary we get that $(\ell_i)_{i \in F \setminus \min(F)} \in S_\alpha(N)$. □

Proof of theorem 3.2 Let (x_i) $\Delta_{(e_i)}$- and Δ-stabilize $\bar{\gamma}$ and γ respectively and let $\alpha < \omega_1$. Since S_α is spreading, $\bar{\gamma} \leq \gamma$. Let $\varepsilon > 0$ and choose $(y_i) \prec (x_i)$ so that for all $(z_i) \prec (y_i)$,

$$| \delta_\alpha(z_i) - \gamma_\alpha | < \varepsilon.$$

For $i \in \mathbb{N}$ set $n_i = \min(\mathrm{ran}\,(y_i))$ w.r.t. (e_i) and choose $L = (n_{m_i})$ by proposition 3.3. For $w \in \mathrm{span}\,(y_{m_i})$ if $w = \sum_{i=j}^{\ell} a_i y_{m_i}$ where $a_j \neq 0$ we set $\bar{w} = \sum_{i=j+1}^{\ell} a_i y_{m_i}$.

Claim: If $(w_i)_1^\ell \prec (y_{m_i})$ is α -admissible w.r.t. (e_j) then $(\bar{w}_i)_1^\ell$ is α -admissible w.r.t. (y_j).

Indeed let $m_{k_i} = \min(\mathrm{ran}\,(w_i))$ w.r.t. (y_j). Then $n_{m_{k_i}} = \min(\mathrm{ran}\,(w_i))$ w.r.t. (e_j), and $(n_{m_{k_i}})_1^\ell \in S_\alpha \Rightarrow (n_{m_{k_i+1}})_1^\ell \in S_\alpha((n_j)) \Rightarrow (m_{k_i+1})_1^\ell \in S_\alpha$. Since $m_{k_i+1} \leq \min(\mathrm{ran}\,(\bar{w}_i))$ w.r.t. (y_j), and S_α is spreading the claim follows.

We may assume that $\|y_{m_i}\| = 1$ for all i and that no subsequence of (y_{m_i}) is equivalent to the unit vector basis of c_0 (indeed, if this were false then clearly $\bar{\gamma}_0 = \gamma_0 = 1$ and $\bar{\gamma}_\alpha = \gamma_\alpha = 0$ for all $\alpha \geq 1$). Thus by taking long averages of (y_{m_i}) we may choose $(z_i) \prec (y_{m_i})$ with the property that for all $z \in \mathrm{span}\,(z_i)$

$$\|z - \bar{z}\| < \varepsilon \|\bar{z}\|.$$

By the definition of $\bar{\delta}_\alpha \equiv \delta_\alpha((z_i), (e_i))$ there exists $(w_i)_1^\ell \prec (z_i)$ which is α-admissible w.r.t. (e_j) and satisfies

$$\| \sum_1^\ell w_i \| < (\bar{\delta}_\alpha + \varepsilon) \sum_1^\ell \|w_i\|.$$

By the above claim $(\bar{w}_i)_1^\ell$ is α-admissible w.r.t. (y_j). Furthermore

$$\begin{aligned} \|\sum_1^\ell \bar{w}_i\| &\leq \|\sum_1^\ell w_i\| + \sum_1^\ell \|w_i - \bar{w}_i\| < (\bar{\delta}_\alpha + \varepsilon)\sum_1^\ell \|w_i\| + \sum_1^\ell \varepsilon\|\bar{w}_i\| \\ &< [(\bar{\delta}_\alpha + \varepsilon)(1+\varepsilon) + \varepsilon]\sum_1^\ell \|\bar{w}_i\|. \end{aligned}$$

It follows that $\gamma_\alpha - \varepsilon < \delta_\alpha(y_i) < (\bar{\gamma}_\alpha + \varepsilon)(1+\varepsilon) + \varepsilon$. Since ε is arbitrary we obtain $\gamma_\alpha \leq \bar{\gamma}_\alpha$ and so $\gamma_\alpha = \bar{\gamma}_\alpha$.

To prove that $\Delta(X) = \Delta(X, (e_i))$, let's first show the inclusion $\subseteq$. Let (x_i) Δ-stabilize $\gamma \in \Delta(X)$. We can find $(y_i) \prec (x_i)$ that $\Delta_{(e_i)}$-stabilizes $\bar{\gamma} \in \Delta_{(e_i)}$. But then (y_i) Δ-stabilizes γ, therefore $\gamma = \bar{\gamma}$, thus $\gamma \in \Delta_{(e_i)}$. The inclusion $\supseteq$ is proved similarly. □

4 The space $T(\theta_n, S_n)_{\mathbb{N}}$

If $\theta_n \not\to 0$ or if $\theta_n = 1$ for some n then $T(\theta_n, S_n)_{\mathbb{N}}$ is isomorphic to ℓ_1. Thus we shall confine ourselves to the case where $\sup \theta_n < 1$ and $\theta_n \to 0$. Furthermore we assume that $\theta_n \searrow 0$ and $\theta_{m+n} \geq \theta_n \theta_m$ for all $n, m \in \mathbb{N}$. Indeed it is easy to see that $T(\theta_n, S_n)_{\mathbb{N}}$ is naturally isometric to $T(\bar{\theta}_n, S_n)_{\mathbb{N}}$ where

$$\bar{\theta}_n \equiv \sup\{\prod_{i=1}^\ell \theta_{k_i} : \sum_{i=1}^\ell k_i \geq n\}.$$

Definition 4.1 *A sequence (θ_n) of scalars is called regular if $(\theta_n) \subset (0,1)$, $\theta_n \searrow 0$ and $\theta_{n+m} \geq \theta_n \theta_m$ for all $n, m \in \mathbb{N}$. If the sequence (θ_n) is regular we define the space $T(\theta_n, S_n)_{\mathbb{N}}$ to be regular.*

Throughout this section, the spaces $T(\theta_n, S_n)_{\mathbb{N}}$ will always be assumed to be regular.

It is easy to see (eg [OTW]) that if a sequence $(b_n) \subset (0,1]$ satisfies $b_{n+m} \geq b_n b_m$ for all $n, m \in \mathbb{N}$ then $\lim_n b_n^{1/n}$ exists and equals $\sup_n b_n^{1/n}$. Therefore, if the sequence (θ_n) is regular then the limit $\theta \equiv \lim_{n\to\infty} \theta_n^{1/n} = \sup_n \theta_n^{1/n}$ exists. Note also that if $(X, \|\!|\cdot|\!\|)$ is a Banach space with a basis, then $\delta_{n+m}(X) \geq \delta_n(X)\delta_m(X)$ for all $n, m \in \mathbb{N}$, thus $\lim_n \delta_n(X)^{1/n} = \sup_n \delta_n(X)^{1/n}$ exists. Furthermore, if X does not contain ℓ_1 isomorphically, then $1 > \delta_n(X) \searrow 0$.

For $n \in \mathbb{N}$, define $\phi_n \equiv \frac{\theta_n}{\theta^n}$. We easily see

- If $\theta = 1$ then $\phi_n = \theta_n \searrow 0$.
- $\phi_{n+m} \geq \phi_n \phi_m$ for all $n, m \in \mathbb{N}$.
- $\phi_n^{1/n} \to 1$.
- $\phi_n \leq 1, \forall n \in \mathbb{N}$.

¿From now on, for a regular sequence (θ_n) we will be referring to the limit $\theta = \lim \theta_n^{1/n}$ and the representation $\theta_n = \theta^n \phi_n$ as above.

The main theorem in this section is the following

Theorem 4.2 *Let $X = T(\theta_n, S_n)_{\mathbb{N}}$ be regular and let $\theta = \lim_n \theta_n^{1/n}$. Then*

(1) For all $Y \prec X$, $\ddot{\delta}_1(Y) = \theta$. Moreover for all $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X so that $\delta_1((X, |\cdot|), (e_i)) > \theta - \varepsilon$.

(2) For all $Y \prec X$ and for all $n \in \mathbb{N}$, $\ddot{\delta}_n(Y) = \theta^n$ and $\ddot{\delta}_\omega(Y) = 0$.

(3) For all $Y \prec X, I_\Delta(Y) = \begin{cases} \omega & \text{if } \theta = 1 \\ 1 & \text{if } \theta < 1 \end{cases}$

(4) If $\frac{\theta_n}{\theta^n} \to 0$ then X is arbitrarily distortable.

To prove the above theorem we need the following two results

Proposition 4.3 *Let $X = T(\theta_n, S_n)_{\mathbb{N}}$ be regular. Then for every $\varepsilon > 0$ there is an equivalent 1-unconditional norm $|\cdot|$ on X such that $\delta_1((X, |\cdot|), (e_i)) \geq \theta - \varepsilon$.*

Theorem 4.4 *Let $X = T(\theta_n, S_n)_{\mathbb{N}}$ be regular. Then for all $Y \prec X$ and $j \in \mathbb{N}$ we have*

$$\delta_j(Y) \leq \theta^j \sup_{p \geq j} \phi_p \vee \frac{\theta_j}{\theta_1}.$$

Proof of theorem 4.2
(1) To prove that if $Y \prec X$ then $\ddot{\delta}_1(Y) \leq \theta$ we note that if $\|\!|\cdot|\!\|$ is an equivalent norm on $T(\theta_i, S_i)_{\mathbb{N}}$ then there exists $C \geq 1$ such that $C^{-1}\delta_n(Y) \leq \delta_n(Y, \|\!|\cdot|\!\|) \leq C\delta_n(Y)$ for all $n \in \mathbb{N}$. Let $\delta_n \equiv \delta_n(Y, \|\!|\cdot|\!\|)$. Then since for all n and m, $\delta_{n+m} \geq \delta_n\delta_m$ we have $\lim_n \delta_n^{1/n} = \sup_n \delta_n^{1/n}$ exists. Hence $\delta_1 \leq \lim \delta_n^{1/n} = \lim \delta_n(Y)^{1/n}$, the latter limit existing for the same reason. Now

$$\lim \delta_n(Y)^{1/n} \leq \lim_{n\to\infty} (\theta^n \sup_{p \geq n} \phi_p \vee \frac{\theta_n}{\theta_1})^{1/n} = \theta$$

by theorem 4.4. Thus $\ddot{\delta}_1(Y) \leq \theta$ as was to be proved. The "moreover" part is proposition 4.3 and this completes the proof of $\ddot{\delta}_1(Y) = \theta$.
(2) Since $\ddot{\delta}_1(Y) = \theta$ we obtain $\ddot{\delta}_n(Y) = \theta^n$ from theorem 2.5. By theorem 4.4 we have that for all $\gamma \in \Delta(Y)$ and for all $j \in \mathbb{N}$,

$$\gamma_j \leq \theta^j \sup_{p \geq j} \phi_p \vee \frac{\theta_j}{\theta_1}.$$

Therefore, again by theorem 2.5, for all $\gamma \in \Delta(Y)$, $\gamma_\omega = \lim_{n \in \mathbb{N}} \gamma_n = 0$. Hence, for every equivalent norm $|\cdot|$ on Y, for every $\gamma \in \Delta(Y, |\cdot|)$, $\gamma_\omega = 0$. Since $\ddot{\delta}_\omega(Y) = \sup\{\gamma_\omega : \gamma \in \ddot{\Delta}(Y)\}$ we have $\ddot{\delta}_\omega(Y) = 0$.
(3) Follows immediately from (2)
(4) Let $\lambda > 1$. Choose $n \in \mathbb{N}$ so that $\sup_{p \geq n} \phi_p < \frac{\theta_1}{2\lambda}$. By (1) we can define an equivalent norm $\|\!|\cdot|\!\|$ on X such that

$$\delta_n((X, \|\!|\cdot|\!\|), (e_i)) \geq \delta_1((X, \|\!|\cdot|\!\|), (e_i))^n \geq \frac{\theta^n}{2}.$$

Let $Y \prec X$. By equation (2) of section 2, there exists $C > 0$ such that

$$C\|\!|y|\!\| \leq \|y\| \leq D(Y, \|\!|\cdot|\!\|)C\|\!|y|\!\|, \text{ for all } y \in Y.$$

Therefore by Observation 2.2,

$$D(Y, ||| \cdot |||) \geq \frac{\delta_n(Y, ||| \cdot |||)}{\delta_n(Y, \| \cdot \|)}.$$

Since $\delta_n(Y, ||| \cdot |||) \geq \delta_n((X, ||| \cdot |||), (e_i)) \geq \frac{\theta^n}{2}$, and $\delta_n(Y, \| \cdot \|) \leq \theta^n \sup_{p \geq n} \phi_p \vee \frac{\theta_n}{\theta_1} \leq \frac{1}{\theta_1} \theta^n \sup_{p \geq n} \phi_p$ (by theorem 4.4), we obtain $D(Y, ||| \cdot |||) \geq \frac{\theta_1}{2 \sup_{p \geq n} \phi_p} > \lambda$. □

The proof of proposition 4.3 comes from an argument in [OTW]. We recall this argument here.

Sketch of the proof of proposition 4.3 Fix $n \in \mathbb{N}$ such that $\theta_n^{1/n} > \theta - \varepsilon$ and set $a \equiv \theta_n^{1/n}$ For $j \in \mathbb{N}$ and $x \in X$ define

$$| x |_j = \sup\{a^j \sum_1^\ell \|E_i x\| : (E_i x)_1^\ell \text{ is } j\text{-admissible w.r.t. } (e_i)\} \text{ and}$$

$$| x | = \frac{1}{n} \sum_{j=0}^{n-1} | x |_j \ (\text{where } | \cdot |_0 = \| \cdot \|).$$

We claim that $\delta_1((X, | \cdot |), (e_i)) \geq a$. To see this let $e_k \leq x_1 < x_2 < \cdots < x_k$ in X and $x = \sum_{i=1}^k x_i$. For $j = 1, \ldots, n-1$ we have $| x |_j \geq a \sum_{i=1}^k | x_i |_{j-1}$ (by the definitions of $| \cdot |_j$ and $| \cdot |_{j-1}$) and also $| x |_0 \geq a \sum_{i=1}^k | x_i |_{n-1}$ (since $a^n = \theta_n$). Therefore we get $| x | \geq a \sum_{i=1}^k | x_i |$. □

To prove theorem 4.4 we need some norm estimates in $T(\theta_n, S_n)_{\mathbb{N}}$ for certain iterated rapidly increasing averages. Before defining what we mean by this we fix some terminology.

Let E be an interval in $\mathbb{N}$ and $x \in c_{00}$. We say that E *does not split* x if either $E \cap \operatorname{ran}(x) = \emptyset$ or $\operatorname{ran}(x) \subseteq E$. Let (x_i) be a block basis of (e_i) in c_{00}, $x \in \operatorname{span}(x_i)_i$, $N \in \mathbb{N}$, and $E_1 < E_2 < \cdots < E_N$ be intervals in $\mathbb{N}$ so that $\cup_{i=1}^N E_i \subseteq \operatorname{ran}(x)$. We say that we *minimally shrink the intervals* $(E_\ell)_{\ell=1}^N$ *to obtain intervals* $(F_\ell)_{\ell=1}^n$ *which don't split the* x_i*'s*, if for $\ell = 1, \ldots, N$ we let $G_\ell = E_\ell \setminus \cup\{\operatorname{ran}(x_i) : E_\ell \text{ splits } x_i\}$ and let $F_1 < F_2 < \cdots F_n$ be the enumeration of the non-empty G_ℓ's.

By a *tree* we shall mean a non-empty partially ordered set $(\mathcal{T}, \ll)$ for which the set $\{y \in \mathcal{T} : y \ll x\}$ is linearly ordered and finite for each $x \in \mathcal{T}$. If $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $(\mathcal{T}', \ll)$ is a *subtree* of $(\mathcal{T}, \ll)$. The tree $\mathcal{T}$ is called *finite* if the set $\mathcal{T}$ is finite. The *initial* nodes of $\mathcal{T}$ are the minimal elements of $\mathcal{T}$ and the *terminal* nodes are the maximal elements. A *branch* in $\mathcal{T}$ is a maximal linearly ordered set in $\mathcal{T}$. The *immediate successors* of $x \in \mathcal{T}$ are all the nodes $y \in \mathcal{T}$ such that $x \ll y$ but there is no $z \in \mathcal{T}$ with $x \ll z \ll y$. If X is a linear space, then a *tree in* X is a tree whose nodes are vectors in X. If X is a Banach space with a basis (e_i) and $(x_i) \prec (e_i)$ then an *admissible averaging tree of* (x_i), is a finite tree $\mathcal{T}$ in X with the following properties:

- $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ where $M \in \mathbb{N}$ and $1 = N^M \leq \cdots \leq N^1 \leq N^0$.
- $x_1^j < \cdots < x_{N^j}^j$ w.r.t. (e_s) $(j = 0, 1, \ldots, M-1)$ & $(x_i^0)_{i=1}^{N^0}$ is a subsequence of (x_s).

Also for $j = 1, \ldots, M$ and $i = 1, \ldots, N^j$ we have the following:

- There exists a non-empty interval $I_i^j \subseteq \{1, \ldots, N^{j-1}\}$ such that $\{x_s^{j-1} : s \in I_i^j\}$ are the immediate successors of x_i^j.
- $x_i^j = \frac{1}{|I_i^j|} \sum_{s \in I_i^j} x_s^{j-1}$.

- $(\min(\mathrm{ran}\,(x_s^{j-1})))_{s\in I_i^j} \in S_1$ where $\mathrm{ran}\,(x_s^{j-1})$ is taken w.r.t. (x_s).

Note that the last two properties together require that x_i^j be a 1-admissible average of all of its immediate successors w.r.t. (x_s). Let $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ be an admissible averaging tree as in the above definition, and let $b = \{y_M \ll \cdots \ll y_0\}$ be a branch in $\mathcal{T}$. For $i = 0, 1, \ldots, M$ we say that the *level* of y_i is i. Note that this is well defined, since the definition of admissible averaging trees forces every branch to have the same number of elements. Indeed for each i and j, the level of x_i^j in $\mathcal{T}$ is j. Let $\mathcal{T}$ be a tree, $x \in \mathcal{T}$ of level ℓ and $k \in \mathbb{N}$. By $\mathcal{T}(x,k)$ (resp. $\mathcal{T}^*(x,k)$) we shall denote the subtree of $\mathcal{T}' = \{x\} \cup \{y \in \mathcal{T} : y \gg x\}$ (resp. $\mathcal{T}' = \{y \in \mathcal{T} : y \gg x\}$) that contains all the nodes of $\mathcal{T}'$ that have level $\ell, \ell-1, \ldots,$ or $\ell-k+1$ in $\mathcal{T}$. Let $\mathcal{T}$ be an admissible averaging tree in a Banach space X with a basis (e_i), $x \in \mathcal{T}$ with immediate successors $x_1 < \cdots < x_n$ (a finite block basis of (e_i)), $k \in \mathbb{N}$, and let $F \subseteq \mathbb{N}$ be an interval which does not split any of $x_1, \ldots, x_n$. Then by $\mathcal{T}_F(x,k)$ we shall denote the subtree of $\mathcal{T}(x,k)$ given by $\mathcal{T}_F(x,k) = \{x\} \cup \{y \in \mathcal{T}^*(x,k) : \mathrm{ran}\,(y) \text{ (w.r.t. } (e_i)) \subseteq F\}$.

Definition 4.5 *Let (x_i) be a block sequence of (e_i) in c_{00}, $M, N \in \mathbb{N}$, and let $(\varepsilon_i^j)_{j,i\in\mathbb{N}} \subset (0,1)$. We say that x is an $(M, (\varepsilon_i^j), N)$ average of (x_i) w.r.t. (e_i) if there exists an admissible averaging tree $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ of (x_i) whose initial node is $x(= x_1^M)$ and*

> *for $j = 1, \ldots, M$ and $1 \le i \le N^j$ if $N_i^j = \max(\mathrm{ran}\,(x_i^j))$ w.r.t (e_s) ($N_0^j = N$), then x_i^j is an average of its immediate successors of length $k_i^j > \frac{2N_{i-1}^j}{\varepsilon_i^j}$.*

$\mathcal{T}$ then will be called an $(M, (\varepsilon_i^j), N)$ admissible averaging tree of (x_i) w.r.t. (e_i). For $i = 1, \ldots, N^0$ set $N_i^0 = \max(\mathrm{ran}\,(x_i^0))$ w.r.t. (e_s), and $N_0^0 = N$. Then $(N_i^j)_{j=0,i=0}^{M,N^j}$ are called the maximum coordinates of $\mathcal{T}$ w.r.t. (e_i).

Remark 4.6 *Let X be a Banach space with basis (e_i) and let (x_i) be a block sequence of (e_i) with $\|x_i\| \le 1$ for all $i \in \mathbb{N}$. Let $(\varepsilon_i^j)_{j,i\in\mathbb{N}} \subset (0,1)$. Let $M, N \in \mathbb{N}$ and let x be an $(M, (\varepsilon_i^j), N)$ average of (x_i) w.r.t. (e_i) given by $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$. Then we can write $x = \sum_{i\in F} a_i x_i$ for some finite set $F \subset \mathbb{N}$ such that*

(1) $\sum_{i\in F} a_i = 1$ & $a_i > 0$ for all $i \in F$.

(2) x is M-admissible w.r.t. (x_i) (i.e. $F \in S_M$).

(3) Let $(N_i^j)_{j=0,i=0}^{M,N^j}$ be the maximum coordinates of $\mathcal{T}$ w.r.t. (e_s). For $j = 1, \ldots, M$ and $1 \le i \le N^j$, let $E_i^j(1) < E_i^j(2) < \cdots < E_i^j(N_{i-1}^j)$ be a finite sequence of intervals in $\mathbb{N}$ with $\cup_{\ell=1}^{N_{i-1}^j} E_i^j(\ell) \subseteq \mathrm{ran}\,(x_i^j)$ and assume that we minimally shrink the $E_i^j(\ell)$'s to obtain intervals $(F_i^j(\ell))_{\ell=1}^{N_{i-1}^j}$ (some of which may be empty) which don't split the x_i^{j-1}'s. Then

$$\sum_{j=1}^{M}\sum_{i=1}^{N^j}\sum_{\ell=1}^{N_{i-1}^j} \|(E_i^j(\ell)\backslash F_i^j(\ell))x_i^j\| < \sum_{j,i} \varepsilon_i^j.$$

Indeed *(1)* and *(2)* are obvious. To see *(3)* note that for every $j = 1, \ldots, M$, $1 \le i \le N^j$ and $\ell = 1, \ldots, N_{i-1}^j$, the set $E_i^j(\ell)$ splits at most two x_s^{j-1}'s each of them having norm at most 1. Thus $\|(E_i^j(\ell)\backslash F_i^j(\ell))x_i^j\| \le 2/k_i^j$ and so $\sum_{\ell=1}^{N_{i-1}^j} \|E_i^j(\ell)\backslash F_i^j(\ell)\| < 2N_{i-1}^j/k_i^j < \varepsilon_i^j$, which proves *(3)*.

The concept of $(M, (\varepsilon_i^j), N)$ vectors is implicit in [AD] (see also [OTW]).

Proposition 4.7 *Let (x_i) be a block sequence in c_{00}, $M,N\in\mathbb{N}$ and $(\varepsilon_i^j)_{j,i\in\mathbb{N}}\subset(0,1)$. Then there exists x which is an $(M,(\varepsilon_i^j),N)$ average of (x_i) w.r.t. (e_i).*

Proof Note that by replacing each $(\varepsilon_i^j)_i$ by a smaller sequence if necessary we may assume that $(\varepsilon_i^j)_i$ is decreasing. For $M=1$ we choose x_1^1 to be an average of $k_1^1>2N/\varepsilon_1^1$ many x_s's chosen from $\{x_s: s\geq k_1^1\}$. Next, consider the case $M=2$. At first we continue the argument that we gave for $M=1$ to construct $\bar{x}_1^1<\bar{x}_2^1<\cdots$ as follows: For $\bar{k}_1^1>2N/\varepsilon_1^1$ let $\bar{x}_1^1$ be an average of $\bar{k}_1^1$ many x_s's chosen from $\{x_s: s\geq \bar{k}_1^1\}$. If $\bar{x}_i^1$ has been constructed for some $i\in\mathbb{N}$, and $\bar{k}_{i+1}^1>2\bar{N}_i^1/\varepsilon_{i+1}^1$, then $\bar{x}_{i+1}^1$ is taken to be an average of $\bar{k}_{i+1}^1$ many x_s's chosen from $\{x_s: s\geq\bar{k}_{i+1}^1\}$ where $\bar{N}_i^1=\max(\mathrm{ran}\,(\bar{x}_i^1))$ w.r.t. (e_s). Note that $\bar{x}_i^1<\bar{x}_{i+1}^1$ since $\varepsilon_{i+1}^1<1$. Also note that for every $i\in\mathbb{N}$, $\bar{x}_i^1$ is a 1-admissible w.r.t. (x_s). Then for $k_1^2>2N/\varepsilon_1^2$ take x_1^2 to be an average of k_1^2 many $\bar{x}_s^1$'s chosen from $\{\bar{x}_s^1:\bar{x}_s^1\geq x_{k_1^2}\}$. Then the $(2,(\varepsilon_i^j),N)$ admissible averaging tree $\mathcal{T}$ of (x_i) that corresponds to x_1^2 is determined as follows: $x_1^2\in\mathcal{T}$. If $x_1^2=\frac{1}{|F|}\sum_{i\in F}\bar{x}_i^1$ for some finite set $F\subset\mathbb{N}$ then $\bar{x}_i^1\in\mathcal{T}$ for $i\in F$. For each $i\in F$ if $\bar{x}_i^1=\frac{1}{|F_i|}\sum_{s\in F_i}x_s$ for some finite set $F_i\subset\mathbb{N}$ then $x_s\in\mathcal{T}$ for $s\in F_i$. Enumerate the x_s's in $\mathcal{T}$ as $x_1^0<x_2^0<\cdots<x_{N^0}^0$ and the $\bar{x}_s^1$'s in $\mathcal{T}$ as $x_1^1<x_2^1<\cdots<x_{N^1}^1$. Since x_1^2 is a 1-admissible average of (x_i^1) w.r.t. (x_i) and for each $i=1,\ldots,N^1$, x_i^1 is 1-admissible w.r.t. (x_s), we have that x_1^2 is 2-admissible w.r.t. (x_i). We let the k_i^1's and N_i^1's be defined by definition 4.5. Each k_i^1 will be $\bar{k}_{i'}^1$ for Some $i'\geq i$ and $N_0^1=N$ while $N_i^1=N_{i'}^1$. Since (ε_i^1) is decreasing the condition $k_i^1>2N_{i-1}^1/\varepsilon_i^1$ remains valid. The case $M>2$ is proved by iterating this procedure. □

Remark 4.8 *Definition 4.5 requires only that $k_i^j>2N_{i-1}^j/\varepsilon_i^j$ The proof shows that we could also construct (x_i^j) so that $k_i^j>\frac{6(N_{i-1}^j)^2}{\theta_1\varepsilon_i^j}$. We will use this remark in lemma 4.12.*

Next we prove some norm estimates for $(M,(\varepsilon_i^j),N)$ averages in $T(\theta_n,S_n)_{\mathbb{N}}$. $\|\cdot\|$ will always denote the norm of $T(\theta_n,S_n)_{\mathbb{N}}$. We need for $p\in\mathbb{N}\cup\{0\}$ and $N\in\mathbb{N}$ to define the equivalent norms $\|\cdot\|_p$ and $\|\cdot\|_{S_N,p}$ and the continuous seminorms $\|\cdot\|_{N,p}$ as follows ($\|\cdot\|_0=\|\cdot\|$ and $\theta_0=1$):

$$\|x\|_p=\theta_p\sup\{\sum\|E_ix\|:(E_i)\text{ is a }p\text{-admissible sequence of intervals }\}$$
$$\|x\|_{N,p}=\sup\{\sum_1^N\|E_ix\|_p: N\leq E_1<E_2<\cdots<E_N\text{ are intervals }\}\text{ and}$$
$$\|x\|_{S_N,p}=\sup\{\sum\|E_ix\|_p:(E_i)\text{ is an }N\text{-admissible sequence of intervals }\}.$$

Of course for $x\in c_{00}$ each "sup" above is a "max" and there exists $p\in\mathbb{N}$ so that $\|x\|=\|x\|_p$ if $\|x\|\neq\|x\|_\infty$.

Remark 4.9 *Let $\theta_0=1$. For all $x\in c_{00}$ and for all $p\in\mathbb{N}$ we have*

$$\|x\|_p\leq\frac{\theta_p}{\theta_{p-1}}\|x\|_{S_1,p-1}.$$

Moreover if $p=1$ we have equality.

Indeed there exists $(E_i)_{i\in I}$ a p-admissible family of intervals such that

$$\|x\|_p=\theta_p\sum_{i\in I}\|E_ix\|.$$

We can write $I = \cup_1^\ell I_j$ where $(E_i)_{i\in I_j}$ is $p-1$-admissible and if F_j is the smallest interval including $\cup_{i\in I_j} E_i$ then $(F_j)_1^\ell$ is 1-admissible. Thus

$$\|x\|_p = \frac{\theta_p}{\theta_{p-1}} \sum_{j=1}^{\ell} \theta_{p-1} \sum_{i\in I_j} \|E_i x\| \le \frac{\theta_p}{\theta_{p-1}} \sum_{j=1}^{\ell} \|F_j x\|_{p-1} \le \frac{\theta_p}{\theta_{p-1}} \|x\|_{S_1,p-1}. \qquad \square$$

Notation If $A \subset [0,\infty)$ is a finite non-empty set, we set $A^* = A\backslash\{\max(A)\}$.

Observation 4.10 *Let $N \in \mathbb{N}$ and $D, \varepsilon > 0$. Note that if $k \ge \frac{ND}{\varepsilon}$ and $A_\ell \subset [0, D]$ for $\ell = 1, \dots, N$ are finite sets with $|A_1| + \cdots + |A_N| \le k$ then $\frac{1}{k}\sum_{\ell=1}^N \sum\{a : a \in A_\ell\} \le \max(\cup_{\ell=1}^N A_\ell^*) + \varepsilon$.*

We will apply this for $D = \frac{1}{\theta_1}$ in the proof of *(2)* of lemma 4.11 below.

Lemma 4.11 *Let non-zero vectors $k \le x_1 < x_2 < \ldots < x_k$ with $\|x_i\| \le 1$ for all i, $x = \frac{1}{k}(x_1 + \cdots + x_k)$ and $\varepsilon \in (0,1)$. Let $F \subseteq \mathrm{ran}\,(x)$ be an interval in $\mathbb{N}$ which does not split the x_i's. Set $\theta_0 = 1$, $N_i = \max(\mathrm{ran}\,(x_i))$ w.r.t. (e_j), $N_0 = 1$ and let $N \in \mathbb{N}$. If $k > \frac{6N}{\theta_1\varepsilon}$ then*

(1) For every $p \in \mathbb{N}$, $\|Fx\|_{N,p} \le \frac{\theta_p}{\theta_{p-1}} \max\{\|x_i\|_{N_{i-1},p-1} : \mathrm{ran}\,(x_i) \subseteq F\} + \varepsilon$.

(2) There exists $n \in \mathbb{N}$, intervals $F_1 < F_2 < \ldots < F_n$ which don't split any x_i, $\cup_{\ell=1}^n F_\ell \subseteq \mathrm{ran}\,(x)$, and $(p_\ell)_{\ell=1}^n \subset \mathbb{N}$ so that

$$\|x\|_{N,0} \le \max\left(\bigcup_{\ell=1}^{n} \{ \frac{\theta_{p_\ell}}{\theta_{p_\ell -1}} \|x_i\|_{N_{i-1},p_\ell -1} : \mathrm{ran}\,(x_i) \subset F_\ell \}^* \right) + \varepsilon.$$

Proof (1) For $p \in \mathbb{N}$ there exist intervals $N \le E_1 < \ldots < E_N$ such that $\cup_{\ell=1}^N E_\ell \subseteq F$ and

$$\|Fx\|_{N,p} = \sum_{\ell=1}^{N} \|E_\ell x\|_p \le \frac{\theta_p}{\theta_{p-1}} \sum_{\ell=1}^{N} \|E_\ell x\|_{S_1,p-1} \text{ (by Remark 4.9)} .$$

We minimally shrink the intervals $(E_i)_1^N$ to get $n \le N$ and intervals $N \le F_1 < F_2 < \cdots < F_n$ which don't split the x_i's. Since each E_ℓ splits at most two x_i's, $\|\cdot\|_{S_1,p-1} \le \frac{1}{\theta_1}\|\cdot\|$ and $\frac{\theta_p}{\theta_{p-1}} \le 1$,

$$\frac{\theta_p}{\theta_{p-1}} \sum_{\ell=1}^{N} \|E_\ell x\|_{S_1,p-1} \le \frac{\theta_p}{\theta_{p-1}} \sum_{\ell=1}^{n} \|F_\ell x\|_{S_1,p-1} + \frac{2N}{k\theta_1}.$$

Fix an $\ell \in \{1, \dots, n\}$. There exists a 1-admissible family of intervals $(F_{\ell,m})_m$ with $F_{\ell,m} \subseteq F_\ell$ for all m and $\|F_\ell x\|_{S_1,p-1} = \sum_m \|F_{\ell,m} x\|_{p-1}$. Let s be minimal with $\mathrm{ran}\,(x_s) \cap F_{\ell,1} \ne \emptyset$ (we may assume that such an s exists) and t be maximal with $\mathrm{ran}\,(x_t) \cap F_\ell \ne \emptyset$. Then

$$\begin{aligned} \sum_m \|F_{\ell,m} x\|_{p-1} &\le \frac{1}{k}(\sum_m \|F_{\ell,m} x_s\| + \|x_{s+1}\|_{N_s,p-1} + \cdots + \|x_t\|_{N_s,p-1}) \\ &\le \frac{1}{k}(\frac{1}{\theta_1} + \sum_{i=s+1}^{t} \|x_i\|_{N_{i-1},p-1}). \end{aligned}$$

Set $[r,R] \equiv \{i : \operatorname{ran}(x_i) \subseteq F\}$. Hence

$$\frac{\theta_p}{\theta_{p-1}} \sum_{\ell=1}^{n} \|F_\ell x\|_{S_1,p-1} \leq \frac{\theta_p}{\theta_{p-1}} \frac{1}{k}\Big(\frac{n}{\theta_1} + \|x_{r+1}\|_{N_r,p-1} + \|x_{r+2}\|_{N_{r+1},p-1} + \cdots + \|x_R\|_{N_{R-1},p-1}\Big).$$

Therefore we have proved that

$$\|Fx\|_{N,p} \leq \frac{1}{k}\frac{\theta_p}{\theta_{p-1}}\Big(\|x_{r+1}\|_{N_r,p-1} + \|x_{r+2}\|_{N_{r+1},p-1} + \cdots + \|x_R\|_{N_{R-1},p-1}\Big) + \frac{3N}{k\theta_1}.$$

Thus

$$\|Fx\|_{N,p} \leq \frac{1}{k}\frac{\theta_p}{\theta_{p-1}} \sum\{\|x_i\|_{N_{i-1},p-1} : \operatorname{ran}(x_i) \subseteq F\} + \frac{3N}{k\theta_1}. \tag{3}$$

This yields (1).

(2) Choose intervals $N \leq E_1 < E_2 < \ldots < E_N$ such that $\|x\|_{N,0} = \sum_{l=1}^{N} \|E_\ell x\|$. As before, we minimally shrink the intervals (E_i) to obtain $n \leq N$ and non-empty intervals $F_1 < F_2 < \cdots < F_n$ which don't split the x_i's and satisfy

$$\sum_{\ell=1}^{N} \|E_\ell x\| \leq \sum_{\ell=1}^{n} \|F_\ell x\| + \frac{2N}{k}.$$

Fix $\ell \in \{1,\ldots,n\}$. If $\|F_\ell x\| \neq \|F_\ell x\|_\infty$ there exists $p_\ell \in \mathbb{N}$ such that $\|F_\ell x\| = \|F_\ell x\|_{p_\ell}$. By equation (3) for $N=1$ we get

$$\|F_\ell x\|_{p_\ell} \leq \frac{1}{k}\frac{\theta_{p_\ell}}{\theta_{p_\ell -1}}\Big(\sum\{\|x_i\|_{N_{i-1},p_\ell-1} : \operatorname{ran}(x_i) \subseteq F_\ell\}\Big) + \frac{3}{k\theta_1}. \tag{4}$$

If $\|F_\ell x\| = \|F_\ell x\|_\infty$ then $\|F_\ell x\| \leq \frac{1}{k}$ and so (4) still is valid. Thus

$$\begin{aligned} \|x\|_{N,0} &\leq \frac{1}{k}\sum_{\ell=1}^{n} \frac{\theta_{p_\ell}}{\theta_{p_\ell-1}} \sum\{\|x_i\|_{N_{i-1},p_\ell-1} : \operatorname{ran}(x_i) \subseteq F_\ell\} + \frac{5N}{k\theta_1} \\ &< \max\left(\bigcup_{\ell=1}^{n}\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}}\|x_i\|_{N_{i-1},p_\ell-1} : \operatorname{ran}(x_i) \subset F_\ell\}^*\right) + \varepsilon \end{aligned}$$

by observation 4.10 since $\|\cdot\|_{N_{i-1},p_\ell-1} \leq \frac{1}{\theta_1}\|\cdot\|$, and $k > \frac{6N}{\varepsilon\theta_1} = \frac{N}{(\varepsilon/6)\theta_1}$. ☐

Combining lemma 4.11 with proposition 4.7 and remark 4.8 we obtain

Lemma 4.12 *Let (x_i) be a normalized block sequence in $X = T(\theta_i, S_i)_{\mathbb{N}}$, $M, N \in \mathbb{N}$ and $(\varepsilon_i^j)_{j,i\in\mathbb{N}} \subset (0,1)$. There exists x, an $(M,(\varepsilon_i^j),N)$ average of (x_i) w.r.t. (e_i), so that if $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ is the $(M,(\varepsilon_i^j),N)$ admissible averaging tree of (x_i) with $x = x_1^M$, and $(N_i^j)_{j=0,i=0}^{M,N^j}$ are the maximum coordinates of $\mathcal{T}$ w.r.t. (e_i) then for $j=1,\ldots,M$ and $i=1,\ldots,N^j$ we have the following properties:*

(1) For every $p \in \mathbb{N}$ and every $F \subseteq \operatorname{ran}(x_i^j)$ which does not split any x_s^{j-1} we have

$$\|Fx_i^j\|_{N_{i-1}^j,p} \leq \frac{\theta_p}{\theta_{p-1}} \max\{\|x_s^{j-1}\|_{N_{s-1}^{j-1},p-1} : \operatorname{ran}(x_s^{j-1}) \subseteq F\} + \varepsilon_i^j/N_{i-1}^j.$$

(2) There exists $n \in \mathbb{N}$ and intervals $F_1 < F_2 < \ldots < F_n$ which don't split any x_s^{j-1}, $(\cup_{\ell=1}^n F_\ell \subseteq \mathrm{ran}\,(x_i^j))$ and $(p_\ell)_{\ell=1}^n \subseteq \mathbb{N}$ such that

$$\|x_i^j\|_{N_{i-1}^j,0} \leq \max\left(\bigcup_{\ell=1}^n \{\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}}\|x_s^{j-1}\|_{N_{s-1}^{j-1},p_\ell-1} : ran(x_s^{j-1}) \subseteq F_\ell\}^*\right) + \varepsilon_i^j.$$

Lemma 4.13 *Let (x_i) be a normalized block sequence in $X = T(\theta_i, S_i)_{\mathbb{N}}$, $\varepsilon > 0$, $(\varepsilon_i^j)_{j,i\in\mathbb{N}} \subset (0,1)$ with $\sum_{j,i}\varepsilon_i^j < \varepsilon$ and let x be an $(M,(\varepsilon_i^j),N)$ average of (x_i) w.r.t. (e_i). Let $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ be the $(M,(\varepsilon_i^j),N)$ admissible averaging tree of (x_i) with $x = x_1^M$, let $(N_i^j)_{j=0,i=0}^{M,N^j}$ be the maximum coordinates of $\mathcal{T}$ w.r.t. (e_i) and assume that for $j = 1,\ldots,M$ and $i = 1,\ldots,N^j$ the properties (1) and (2) of lemma 4.12 are satisfied. Then we have*

(3) If $0 \leq p' < p$, $p - p' \leq j \leq M$, $1 \leq i \leq N^j$ and $F \subseteq \mathrm{ran}\,(x_i^j)$ is an interval which does not split any x_s^{j-1} then

$$\|Fx_i^j\|_{N_{i-1}^j,p} \leq \frac{\theta_p}{\theta_{p'}}\max\{\|x_s^{j-(p-p')}\|_{N_{s-1}^{j-(p-p')},p'} : \mathrm{ran}\,(x_s^{j-(p-p')}) \subseteq F\} + \sum_{x_s^k\in\mathcal{T}_F(x_i^j,p-p')}\frac{\varepsilon_s^k}{N_{s-1}^k}.$$

(4) If $1 \leq p \leq j \leq M$, $1 \leq i \leq N^j$ and $F \subseteq \mathbb{N}$ is an interval which does not split any x_s^{j-1} then

$$\|Fx_i^j\|_{N_{i-1}^j,p} \leq \theta_p\max\{\|x_s^{j-p}\|_{N_{s-1}^{j-p},0} : \mathrm{ran}\,(x_s^{j-p}) \subseteq F\} + \sum\{\frac{\varepsilon_s^k}{N_{s-1}^k} : x_s^k \in \mathcal{T}_F(x_i^j,p)\}.$$

(5) There exists $m \in \mathbb{N}$ and intervals $F_1 < F_2 < \ldots < F_m$ $(\cup_\ell F_\ell \subseteq \mathrm{ran}\,(x_1^M))$ which don't split the x_s^0's and $(p_\ell)_{\ell=1}^m \subset \mathbb{N}$ with $p_\ell \geq M$ for all ℓ, such that

$$\|x_1^M\| \leq \max\left(\bigcup_{\ell=1}^m \{\frac{\theta_{p_\ell}}{\theta_{p_\ell-M}}\|x_s^0\|_{N_{s-1}^0,p_\ell-M} : \mathrm{ran}\,(x_s^0) \subseteq F_\ell\}^*\right) + \varepsilon.$$

Proof
(3) By (1) of lemma 4.12 we have

$$\begin{aligned}
&\|Fx_i^j\|_{N_{i-1}^j,p} \leq \frac{\theta_p}{\theta_{p-1}}\max\{\|x_s^{j-1}\|_{N_{s-1}^{j-1},p-1} : \mathrm{ran}\,(x_s^{j-1}) \subseteq F\} + \frac{\varepsilon_i^j}{N_{i-1}^j}\\
\leq\; & \frac{\theta_p}{\theta_{p-1}}\frac{\theta_{p-1}}{\theta_{p-2}}\max\{\|x_s^{j-2}\|_{N_{s-1}^{j-2},p-2} : \mathrm{ran}\,(x_s^{j-2}) \subseteq F\} + \sum\{\frac{\varepsilon_s^k}{N_{s-1}^k} : x_s^k \in \mathcal{T}_F(x_i^j,2)\}\\
\leq\; & \ldots\\
\leq\; & \frac{\theta_p}{\theta_{p-1}}\frac{\theta_{p-1}}{\theta_{p-2}}\ldots\frac{\theta_{p'+1}}{\theta_{p'}}\max\{\|x_s^{j-(p-p')}\|_{N_{s-1}^{j-(p-p')},p'} :\\
&\qquad \mathrm{ran}\,(x_s^{j-(p-p')}) \subseteq F\} + \sum\{\frac{\varepsilon_s^k}{N_{s-1}^k} : x_s^k \in \mathcal{T}_F(x_i^j,p-p')\}.
\end{aligned}$$

(4) Follows immediately from (3), letting $p' = 0$.
(5) We prove by induction on J that

for $J=1,\dots,M$ and $1\le i\le N^J$ there exists $m\in\mathbb{N}$, intervals $F_1<F_2<\cdots<F_m$ $(\cup_\ell F_\ell\subseteq \operatorname{ran}(x_i^J))$ that don't split the x_s^0's, and $(p_\ell)_{\ell=1}^m\subset\mathbb{N}$ with $p_\ell\ge J$ for all ℓ, such that

$$\|x_i^J\|_{N_{i-1}^J,0}\le \max\left(\bigcup_{\ell=1}^m\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-J}}\|x_s^0\|_{N_{s-1}^0,p_\ell-J}:\operatorname{ran}(x_s^0)\subseteq F_\ell\}^*\right)+\sum\{\varepsilon_s^k:x_s^k\in\mathcal{T}(x_i^J,J)\}$$

((5) then follows by taking $(J,i)=(M,1)$ and noting that $\|x_1^M\|\le\|x_1^M\|_{N,0}=\|x\|_{N_0^M,0}$). Indeed, for $J=1$ this follows from the statement of (2) for $j=1$. Assume that the statement is proved for all positive integers $\le J$ where $J\le M-1$. By (2) there exist intervals $F_1'<\cdots<F_n'$ $(\cup_\ell F_\ell'\subseteq\operatorname{ran}(x_i^{J+1}))$ which don't split the x_s^J's, and $(p_\ell')_{\ell=1}^n$ such that

$$\|x_i^{J+1}\|_{N_{i-1}^{J+1},0}\le\max\left(\bigcup_{\ell=1}^n\{\frac{\theta_{p_\ell'}}{\theta_{p_\ell'-1}}\|x_s^J\|_{N_{s-1}^J,p_\ell'-1}:\operatorname{ran}(x_s^J)\subseteq F_\ell'\}^*\right)+\varepsilon_i^{J+1}.$$

If $p_\ell'-1=0$ for some ℓ and $\operatorname{ran}(x_s^J)\subseteq F_\ell'$ then by the induction hypothesis there exists $M(s)\in\mathbb{N}$, intervals $F_1(s)<F_2(s)<\cdots<F_{M(s)}(s)$ $(\cup_\mu F_\mu(s)\subseteq\operatorname{ran}(x_s^J))$ that don't split the x_t^0's and $(p_\mu(s))_{\mu=1}^{M(s)}\subset\mathbb{N}$ with $p_\mu(s)\ge J$ for all μ such that

$$\|x_s^J\|_{N_{s-1}^J,0}\le\max\left(\bigcup_{\mu=1}^{M(s)}\{\frac{\theta_{p_\mu(s)}}{\theta_{p_\mu(s)-J}}\|x_t^0\|_{N_{t-1}^0,p_\mu(s)-J}:\operatorname{ran}(x_t^0)\subseteq F_\mu(s)\}^*\right)+\sum\{\varepsilon_t^k:x_t^k\in\mathcal{T}(x_s^J,J)\}.$$

If $0<p_\ell'-1\le J$ for some ℓ, and $\operatorname{ran}(x_s^J)\subseteq F_\ell'$ then by (4),

$$\|x_s^J\|_{N_{s-1}^J,p_\ell'-1}\le\theta_{p_\ell'-1}\max\{\|x_t^{J-p_\ell'+1}\|_{N_{t-1}^{J-p_\ell'+1},0}:\operatorname{ran}(x_t^{J-p_\ell'+1})\subseteq\operatorname{ran}(x_s^J)\}+\sum_{x_t^k\in\mathcal{T}(x_s^J,p_\ell'-1)}\varepsilon_t^k.$$

For the remaining ℓ's we have by (3) for $j=J$, $p=p_\ell'-1$ and $p'=p_\ell'-1-J$,

$$\|x_s^J\|_{N_{s-1}^J,p_\ell'-1}\le\frac{\theta_{p_\ell'-1}}{\theta_{p_\ell'-1-J}}\max\{\|x_t^0\|_{N_{t-1}^0,p_\ell'-1-J}:\operatorname{ran}(x_t^0)\subseteq\operatorname{ran}(x_s^J)\}+\sum\{\varepsilon_t^k:x_t^k\in\mathcal{T}(x_s^J,J)\}.$$

Combining these estimates we get

$$\begin{aligned}
&\|x_i^{J+1}\|_{N_{i-1}^{J+1},0}\le\\
&\max\Bigg(\bigcup_{\{\ell:p_\ell'=1\}}\ \bigcup_{\{s:\operatorname{ran}(x_s^J)\subseteq F_\ell'\}}\ \bigcup_{\mu=1}^{M(s)}\{\frac{\theta_1\theta_{p_\mu(s)}}{\theta_{p_\mu(s)-J}}\|x_t^0\|_{N_{t-1}^0,p_\mu(s)-J}+\sum\{\varepsilon_w^k:x_w^k\in\mathcal{T}(x_s^J,J)\}:\operatorname{ran}(x_t^0)\subseteq F_\mu(s)\}^*\\
&\cup\bigcup_{\{\ell:0<p_\ell'-1\le J\}}\{\theta_{p_\ell'}\|x_t^{J-p_\ell'+1}\|_{N_{t-1}^{J-p_\ell'+1},0}+\sum\{\varepsilon_s^k:x_s^k\in\mathcal{T}^*(x_i^{J+1},p_\ell')\}:\operatorname{ran}(x_t^{J-p_\ell'+1})\subseteq F_\ell'\}^*\\
&\cup\bigcup_{\{\ell:p_\ell'>J+1\}}\{\frac{\theta_{p_\ell'}}{\theta_{p_\ell'-(J+1)}}\|x_t^0\|_{N_{t-1}^0,p_\ell'-(J+1)}+\sum\{\varepsilon_s^k:x_s^k\in\mathcal{T}^*(x_i^{J+1},J+1)\}:\operatorname{ran}(x_t^0)\subseteq F_\ell'\}^*\Bigg)+\varepsilon_i^{J+1}.
\end{aligned}$$

The induction hypothesis gives that for $0 < p'_\ell - 1 < J$ and $1 \le t \le N^{J-p'_\ell+1}$ with $\operatorname{ran}(x_t^{J-p'_\ell+1}) \subseteq F'_\ell$, there exists $K(\ell,t) \in \mathbb{N}$ and sets $G_1(\ell,t) < G_2(\ell,t) < \ldots < G_{K(\ell,t)}(\ell,t)$ which don't split the x_s^0's such that $\cup_k G_k(\ell,t) \subseteq \operatorname{ran}(x_t^{J-p'_\ell+1})$, and there exist $(q_k(\ell,t))_{k=1}^{K(\ell,t)} \subset \mathbb{N}$ with $q_k(\ell,t) \ge J - p'_\ell + 1$ such that

$$\begin{aligned}\|x_t^{J-p'_\ell+1}\|_{N_{t-1}^{J-p'_\ell+1},0} &\le \max\left(\bigcup_{k=1}^{K(\ell,t)}\{\frac{\theta_{q_k(\ell,t)}}{\theta_{q_k(\ell,t)-(J-p'_\ell+1)}}\|x_s^0\|_{N_{s-1}^0,q_k(\ell,t)-(J-p'_\ell+1)} : \operatorname{ran}(x_s^0) \subseteq G_k(\ell,t)\}^*\right)\\ &+ \sum\{\varepsilon_s^k : x_s^k \in \mathcal{S}(\ell,t)\}\end{aligned}$$

where $\mathcal{S}(\ell,t) = \mathcal{T}(x_t^{J-p'_\ell+1}, J - p_\ell + 1)$. Thus, these estimates give

$$\begin{aligned}&\|x_i^{J+1}\|_{N_{i-1}^{J+1},0} \le\\ &\max\Bigg(\bigcup_{\{\ell:p'_\ell=1\}}\bigcup_{\{s:\operatorname{ran}(x_s^J)\subseteq F'_\ell\}}\bigcup_{\mu=1}^{M(s)}\{\frac{\theta_1\theta_{p_\mu(s)}}{\theta_{(1+p_\mu(s))-(J+1)}}\|x_t^0\|_{N_{t-1}^0,p_\mu(s)-J}\\ &\qquad + \sum\{\varepsilon_w^k : x_w^k \in \mathcal{T}(x_s^J, J)\} : \operatorname{ran}(x_t^0) \subseteq F_\mu(s)\}^*\\ &\cup \bigcup_{\{\ell:0<p'_\ell-1<J\}}\bigcup_{k=1}^{K(\ell,t)}\{\frac{\theta_{p'_\ell}\theta_{q_k(\ell,t)}}{\theta_{(p'_\ell+q_k(\ell,t))-(J+1)}}\|x_s^0\|_{N_{s-1}^0,q_k(\ell,t)-(J-p'_\ell+1)}\\ &\qquad + \sum\{\varepsilon_s^k : x_s^k \in \mathcal{T}^*(x_i^{J+1}, p'_\ell) \cup \mathcal{S}(\ell,t)\} : \operatorname{ran}(x_s^0) \subseteq G_k(\ell,t)\}^*\\ &\cup \bigcup_{\{\ell:p'_\ell=J+1\}}\{\theta_{J+1}\|x_t^0\|_{N_{t-1}^0,0} + \sum\{\varepsilon_s^k : x_s^k \in \mathcal{T}^*(x_i^{J+1}, J+1)\} : \operatorname{ran}(x_t^0) \subseteq F'_\ell\}^*\\ &\cup \bigcup_{\{\ell:p'_\ell>J+1\}}\{\frac{\theta_{p'_\ell}}{\theta_{p'_\ell-(J+1)}}\|x_t^0\|_{N_{t-1}^0,p'_\ell-(J+1)} + \sum\{\varepsilon_s^k : x_s^k \in \mathcal{T}^*(x_i^{J+1}, J+1)\} : \operatorname{ran}(x_t^0) \subseteq F'_\ell\}^*\Bigg) + \varepsilon_i^{J+1}.\end{aligned}$$

Note that $\theta_1\theta_{p_\mu(s)} \le \theta_{1+p_\mu(s)}$, $1+p_\mu(s) \ge J+1$, $\theta_{p'_\ell}\theta_{q_k(\ell,t)} \le \theta_{p'_\ell+q_k(\ell,t)}$, $p'_\ell+q_k(\ell,t) \ge p'_\ell+(J-p'_\ell+1) = J+1$, the sets $F_\mu(s)$'s F'_ℓ's and $G_k(\ell,t)$'s don't split the x_s^0's, and arranged in successive order, give the required sequence $F_1 < \ldots < F_m$. Then $1 + p_\mu(s)$'s, $p'_\ell + q_k(\ell,t)$'s, and p'_ℓ's for $p'_\ell \ge J+1$ arranged in the corresponding order, give the required sequence $(p_\ell)_{\ell=1}^m$. This finishes the induction. □

Combining lemmas 4.12 and 4.13 we immediately obtain

Corollary 4.14 *Let (x_i) be a normalized block sequence in $X = T(\theta_i, S_i)_\mathbb{N}$, $M, N \in \mathbb{N}$, $\varepsilon > 0$ and $(\varepsilon_i^j)_{j,i\in\mathbb{N}} \subset (0,1)$ with $\sum_{j,i}\varepsilon_i^j < \varepsilon$. There exists x an $(M, (\varepsilon_i^j), N)$ average of (x_i^0) w.r.t. (e_i), so that if $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ is the admissible averaging tree of (x_i) with $x = x_1^M$, and $(N_i^j)_{j=0,i=0}^{M,N^j}$ are the maximum coordinates of $\mathcal{T}$ w.r.t. (e_i), then*

(1) For $j = 1, \ldots, M$, $i = 1, \ldots, N^j$, $1 \le p \le j$ and an interval $F \subseteq \operatorname{ran}(x_i^j)$ which does not split any x_s^{j-1},

$$\|Fx_i^j\|_{N_{i-1}^j,p} \le \theta_p \max\{\|x_s^{j-p}\|_{N_{s-1}^{j-p},0} : \operatorname{ran}(x_s^{j-p}) \subseteq F\} + \sum\{\frac{\varepsilon_s^k}{N_{s-1}^k} : x_s^k \in \mathcal{T}_F(x_i^j, p)\}.$$

(2) There exists $m \in \mathbb{N}$ and intervals $F_1 < F_2 < \ldots < F_m$ which don't split the x_s^0's and $(p_\ell)_{\ell=1}^m \subset \mathbb{N}$ with $p_\ell \geq M$ for all ℓ, such that

$$\|x_1^M\| \leq \max\left(\bigcup_{\ell=1}^m \{\frac{\theta_{p_\ell}}{\theta_{p_\ell - M}} \|x_s^0\|_{N_{s-1}^0, p_\ell - M} : \operatorname{ran}(x_s^0) \subseteq F_\ell\}^*\right) + \varepsilon.$$

To prove theorem 4.4 we need also the following

Lemma 4.15 *For all $J, N \in \mathbb{N}$, $\varepsilon > 0$ and $Y \prec X = T(\theta_i, S_i)_{\mathbb{N}}$ there exists $y \in Y$ with $\|y\| = 1$ and*

$$\|y\|_{N,p} < \phi_p(1+\varepsilon), \text{ for all } p = 1, \ldots J.$$

Proof If this were false, then $\exists J, N \in \mathbb{N}$ $\exists \varepsilon \in (0, 1/2)$ $\exists Y \prec X$ such that

$$\|y\| \leq \max_{1 \leq p \leq J} \frac{1}{\phi_p(1+\varepsilon)} \|y\|_{N,p} \text{ for all } y \in Y. \tag{5}$$

Since $(1+\varepsilon)^n \phi_{J(n+1)} \to \infty$ as $n \to \infty$ we may choose $n \in \mathbb{N}$ such that

$$1 > \frac{1}{(1+\varepsilon)^n \theta^J \theta_1 \phi_{J(n+1)}} + 2\varepsilon.$$

Let (x_s) be a normalized block sequence in Y and apply corollary 4.14 to (x_s) for $(M, \varepsilon, N) = (J(n+1), \varepsilon\theta_{J(n+1)}\phi_1^J, N)$ for an appropriate sequence (ε_i^j), to construct $x = \sum a_s x_s^0$, a $(J(n+1), (\varepsilon_i^j), N)$ average of (x_s) w.r.t. (e_s). Let x have a corresponding admissible averaging tree $\mathcal{T} = (x_i^j)_{j=0,i=1}^{J(n+1),N^j}$, and let the maximum coordinates of $\mathcal{T}$ be $(N_i^j)_{j=0,i=0}^{J(n+1),N^j}$ w.r.t. (e_i). Define $\delta_i^j = \varepsilon_i^j / \phi_1^J$ for $j, i \in \mathbb{N}$ and note that $\sum \delta_i^j < \varepsilon\theta_{J(n+1)}$. Note that if $1 \leq p \leq J$ then $\phi_p \geq \phi_1^p \geq \phi_1^J$ and if $k, s \in \mathbb{N}$ then we have that $\frac{\varepsilon_s^k}{\phi_p} \leq \delta_s^k$. There exists $1 \leq p^1 \leq J$ so that

$$\theta_{J(n+1)} = \theta_{J(n+1)} \sum \|a_s x_s^0\| \leq \|x\| \leq \frac{1}{\phi_{p^1}(1+\varepsilon)} \|x\|_{N,p^1} = \frac{1}{\phi_{p^1}(1+\varepsilon)} \|x_1^{J(n+1)}\|_{N_0^{J(n+1)}, p^1}$$

(since $N = N_0^{J(n+1)}$). Then by corollary 4.14 (1), there exists $s^1 \in \mathbb{N}$ so that $\operatorname{ran}(x_{s^1}^{J(n+1)-p^1}) \subseteq \operatorname{ran}(x_1^{J(n+1)})$ and also there exists a family of intervals $(E_i)_{i=1,\ldots,N_{s^1-1}^{J(n+1)-p^1}} \subseteq \operatorname{ran}(x_1^{J(n+1)})$ so that

$$\begin{aligned}
\theta_{J(n+1)} &\leq \frac{1}{\phi_{p^1}(1+\varepsilon)} \left(\theta_{p^1} \|x_{s^1}^{J(n+1)-p^1}\|_{N_{s^1-1}^{J(n+1)-p^1}, 0} + \sum\{\varepsilon_s^k : x_s^k \in \mathcal{T}(x_1^{J(n+1)}, p^1)\}\right) \\
&\leq \sum_{i=1}^{N_{s^1-1}^{J(n+1)-p^1}} \frac{\theta^{p^1}}{1+\varepsilon} \|E_i x_{s^1}^{J(n+1)-p^1}\| + \sum\{\delta_s^k : x_s^k \in \mathcal{T}(x_1^{J(n+1)}, p^1)\}
\end{aligned}$$

We minimally shrink the E_i's if necessary, to obtain (F_i) which don't split the $x_s^{J(n+1)-p^1-1}$'s. Let $\mathcal{A}$ be the set of $x_s^{J(n+1)-p^1-1}$'s that is split by the E_i's. Thus we get

$$\theta_{J(n+1)} \leq \sum_i \frac{\theta^{p^1}}{1+\varepsilon} \|F_i x_{s^1}^{J(n+1)-p^1}\| + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum\{\delta_s^k : x_s^k \in \mathcal{T}(x_1^{J(n+1)}, p^1)\}.$$

Similarly by 5 for each i there exist $1 \leq p_i^2 \leq J$ so that (note that $N \leq N_{s^1-1}^{J(n+1)-p^1}$)

$$\begin{aligned}&\theta_{J(n+1)} \leq \\ &\sum_i \frac{\theta^{p^1}}{(1+\varepsilon)} \frac{1}{\phi_{p_i^2}(1+\varepsilon)} \|F_i x_{s^1}^{J(n+1)-p^1}\|_{N,p_i^2} + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum\{\delta_s^k : x_s^k \in \mathcal{T}(x_1^{J(n+1)}, p^1)\} \\ \leq\ &\sum_i \frac{\theta^{p^1}}{(1+\varepsilon)} \frac{1}{\phi_{p_i^2}(1+\varepsilon)} \|F_i x_{s^1}^{J(n+1)-p^1}\|_{N_{s^1-1}^{J(n+1)-p^1}, p_i^2} + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum_{x_s^k \in \mathcal{T}(x_1^{J(n+1)}, p^1)} \delta_s^k.\end{aligned}$$

Then by corollary 4.14 (1), for each i there exists $s_i^2 \in \mathbb{N}$ so that $\operatorname{ran}(x_{s_i^2}^{J(n+1)-p^1-p_i^2}) \subseteq F_i$ and also family of intervals $(E_{i,j})_{j=1,\ldots,N_{s_i^2-1}^{J(n+1)-p^1-p_i^2}} \subseteq \operatorname{ran}(x_{s^1}^{J(n+1)-p^1})$ so that

$$\begin{aligned}&\theta_{J(n+1)} \leq \\ &\sum_i \frac{\theta^{p^1}}{(1+\varepsilon)} \frac{1}{\phi_{p_i^2}(1+\varepsilon)} \left(\theta_{p_i^2} \|x_{s_i^2}^{J(n+1)-p^1-p_i^2}\|_{N_{s_i^2-1}^{J(n+1)-p^1-p_i^2}, 0} + \sum\{\frac{\varepsilon_s^k}{N_{s-1}^k} : x_s^k \in \mathcal{T}_{F_i}(x_{s^1}^{J(n+1)-p^1}, p_i^2)\} \right) \\ &+2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum\{\delta_s^k : x_s^k \in \mathcal{T}(x_1^{J(n+1)}, p^1)\} \leq \\ &\sum_i \sum_{j=1}^{N_{s_i^2-1}^{J(n+1)-p^1-p_i^2}} \frac{\theta^{p^1+p_i^2}}{(1+\varepsilon)^2} \|E_{i,j} x_{s_i^2}^{J(n+1)-p^1-p_i^2}\| + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum\{\delta_s^k : x_s^k \in \mathcal{S}\}\end{aligned}$$

where $\mathcal{S} = \mathcal{T}(x_1^{J(n+1)}, p^1) \cup \cup_i \cup \{\mathcal{T}(x_t^{J(n+1)-p^1-1}, p_i^2) : \operatorname{ran}(x_t^{J(n+1)-p^1-1}) \subseteq F_i\}$. We increase $\mathcal{A}$ by including every node $x_s^{J(n+1)-p^1-p_i^2-1}$ which is split by some $E_{i,j}$ and minimally shrink the $E_{i,j}$'s to get intervals $(F_{i,j})$ which don't split the $x_s^{J(n+1)-p^1-p_i^2-1}$'s. Thus

$$\theta_{J(n+1)} \leq \sum_i \sum_j \frac{\theta^{p^1+p_i^2}}{(1+\varepsilon)^2} \|F_{i,j} x_{s_i^2}^{J(n+1)-p^1-p_i^2}\| + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum\{\delta_s^k : x_s^k \in \mathcal{S}\}.$$

For every i, j there exists $1 \leq p_{i,j}^3 \leq J$ so that we have (note also that $N \leq N_{s_i^2-1}^{J(n+1)-p^1-p_i^2}$)

$$\begin{aligned}&\theta_{J(n+1)} \leq \\ &\sum_i \sum_j \frac{\theta^{p^1+p_i^2}}{(1+\varepsilon)^2} \frac{1}{\phi_{p_{i,j}^3}(1+\varepsilon)} \|F_{i,j} x_{s_i^2}^{J(n+1)-p^1-p_i^2}\|_{N,p_{i,j}^3} + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} + \sum\{\delta_s^k : x_s^k \in \mathcal{S}\} \\ \leq\ &\sum_i \sum_j \frac{\theta^{p^1+p_i^2}}{(1+\varepsilon)^2} \frac{1}{\phi_{p_{i,j}^3}(1+\varepsilon)} \|F_{i,j} x_{s_i^2}^{J(n+1)-p^1-p_i^2}\|_{N_{s_i^2-1}^{J(n+1)-p^1-p_i^2}, p_{i,j}^3} + 2\sum\{\|x_s^k\| : x_s^k \in \mathcal{A}\} \\ &+\sum\{\delta_s^k : x_s^k \in \mathcal{S}\}.\end{aligned}$$

By corollary 4.14 (1), for each i, j there exists $s_{i,j}^3 \in \mathbb{N}$ so that

$$\theta_{J(n+1)} \leq \sum_i \sum_j \frac{\theta^{p^1+p_i^2}}{(1+\varepsilon)^2} \frac{1}{\phi_{p_{i,j}^3}(1+\varepsilon)} \Bigg(\theta_{p_{i,j}^3} \|x_{s_{i,j}^3}^{J(n+1)-p^1-p_i^2-p_{i,j}^3}\|_{N_{s_{i,j}^3-1}^{J(n+1)-p^1-p_i^2-p_{i,j}^3}, 0}$$

$$+\sum\{\frac{\varepsilon_s^k}{N_{s-1}^k}:x_s^k\in\mathcal{T}_{F_{i,j}}(x_{s_i^2}^{J(n+1)-p^1-p_i^2},p_{i,j}^3)\}\Big)+2\sum\{\|x_s^k\|:x_s^k\in\mathcal{A}\}+\sum\{\delta_s^k:x_s^k\in\mathcal{S}\}$$

$$\leq\sum_i\sum_j\frac{\theta^{p^1+p_i^2+p_{i,j}^3}}{(1+\varepsilon)^3}\|x_{s_{i,j}^3}^{J(n+1)-p^1-p_i^2-p_{i,j}^3}\|_{N_{s_{i,j}^3-1}^{J(n+1)-p^1-p_i^2-p_{i,j}^3},0}+2\sum\{\|x_s^k\|:x_s^k\in\mathcal{A}\}+\sum\{\delta_s^k:x_s^k\in\mathcal{S}'\}$$

for some $\mathcal{S}'\subseteq\mathcal{T}$. We continue passing to lower levels of the tree until we obtain $Jn\leq p^1+p_i^2+\cdots+p_{i,\dots,k}^r\leq J(n+1)-1$. On each branch of the tree we stop when this is satisfied. Thus we get an estimate of the following form ($\mathcal{A}$ increases to contain the x_s^k's that are split)

$$\begin{aligned}\theta_{J(n+1)} &\leq \sum\sum_{i,\dots,k}\frac{\theta^{p^1+p_i^2+\cdots+p_k^r}}{(1+\varepsilon)^r}\|x_{s_{i,\dots,k}^r}^{J(n+1)-p^1-p_i^2\cdots-p_{i,\dots,k}^r}\|_{N_{s_{i,\dots,k}^r-1}^{J(n+1)-p^1-p_i^2\cdots-p_{i,\dots,k}^r},0}\\ &\quad +2\sum\{\|x_s^k\|:x_s^k\in\mathcal{A}\}+\sum\{\delta_s^k:x_s^k\in\mathcal{W}\}\end{aligned}$$

for some $\mathcal{W}\subseteq\mathcal{T}$, where the first "$\sum$" is taken over all branches on which we have $Jn\leq p^1+p_i^2+\cdots+p_{i,\dots,k}^r\leq J(n+1)-1$. By remark 4.6 (3) we have that $2\sum\{\|x_s^k\|:x_s^k\in\mathcal{A}\}<\varepsilon\theta_{J(n+1)}$. Also $\sum\{\delta_s^k:x_s^k\in\mathcal{W}\}<\varepsilon\theta_{J(n+1)}$. Thus

$$\theta_{J(n+1)}\leq\frac{\theta^{Jn}}{(1+\varepsilon)^n}\sum\sum_{i,\dots,k}\|x_{s_{i,\dots,k}^r}^{J(n+1)-p^1-p_i^2\cdots-p_{i,\dots,k}^r}\|_{N_{s_{i,\dots,k}^r-1}^{J(n+1)-p^1-p_i^2\cdots-p_{i,\dots,k}^r},0}+2\varepsilon\theta_{J(n+1)}.$$

Since $\|\cdot\|_{n,0}\leq\frac{1}{\theta_1}\|\cdot\|$, the vectors $x_{s_{i,\dots,k}^r}^{J(n+1)-p^1-p_i^2\cdots-p_{i,\dots,k}^r}$'s have disjoint support and their level in the tree is at least 1,by the triangle inequality we obtain

$$\theta_{J(n+1)}\leq\frac{\theta^{Jn}}{\theta_1(1+\varepsilon)^n}+2\varepsilon\theta_{J(n+1)}\Leftrightarrow 1\leq\frac{1}{(1+\varepsilon)^n\theta^J\theta_1\phi_{J(n+1)}}+2\varepsilon$$

which is a contradiction. □

Proof of theorem 4.4 Let $\varepsilon>0$ be arbitrary. By lemma 4.15 we can find a normalized block sequence (x_i) in Y and an increasing sequence $(\bar{j}_i)$ of integers, $\bar{j}_1=1$, so that if $N_0=1$ and $N_i=\max(\operatorname{ran}(x_i))$ w.r.t. (e_s) then for every $i\in\mathbb{N}$ we have

$$\begin{aligned}&\forall p=1,\dots,\bar{j}_i, && \|x_i\|_{N_{i-1},p}<\phi_p(1+\varepsilon)\text{ and}\\ &\forall p\geq\bar{j}_{i+1}, && \|x_i\|_{N_{i-1},p}<\varepsilon.\end{aligned}$$

Apply corollary 4.14 for (x_i), ε, $N=1$ and $M=j$ (and appropriate (ε_i^k)) to obtain x, a $(j,(\varepsilon_i^k),1)$ average of (x_i) w.r.t. (e_i) with admissible averaging tree $(x_i^k)_{k=0,i=1}^{j,N^k}$ of (x_i) and maximum coordinates $(N_i^k)_{k=1,i=0}^{j,N^k}$ w.r.t. (e_i). For $i=1,\dots,N^0$ if $x_i^0=x_s$ then define $j_i=\bar{j}_s$. Then $j_1<\cdots<j_{N^0}$ and for $i=1,\dots,N^0$ we have

$$\begin{aligned}&\forall p=1,\dots,j_i, && \|x_i^0\|_{N_{i-1}^0,p}<\phi_p(1+\varepsilon)\text{ and}\\ &\forall p\geq j_{i+1}, && \|x_i^0\|_{N_{i-1}^0,p}<\varepsilon.\end{aligned}$$

Note (by remark 4.6 (2)) that x_1^j is j-admissible w.r.t. (x_i) and by corollary 4.14 (2) there exist $m\in\mathbb{N}$, intervals $F_1<\ldots<F_m$ which don't split the x_s^0's, and $(p_\ell)_{\ell=1}^m\subset\mathbb{N}$ with $p_\ell\geq j$ for all ℓ such that

$$\|x\|\leq\max\left(\bigcup_{\ell=1}^m\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-j}}\|x_s^0\|_{N_{s-1}^0,p_\ell-j}:\operatorname{ran}(x_s^0)\subseteq F_\ell\}^*\right)+\varepsilon.$$

For each $\ell = 1, \ldots, m$ if $p_\ell > j$ then there exists exactly one $m_\ell \in \mathbb{N}$ such that $j_{m_\ell} \le p_\ell - j < j_{m_\ell+1}$. We shall use the obvious remark that if $A \subseteq [0,\infty)$ is a finite non-empty set and $a \in A$ then $\max(A^*) \le \max(A\backslash\{a\})$. If $p_\ell = j$ then $\theta_{p_\ell}/\theta_{p_\ell - j} = \theta_j$ and we note that $\|x_s^0\|_{N_{s-1}^0,0} \le \frac{1}{\theta_1}\|x_s^0\| = \frac{1}{\theta_1}$. Thus

$$\|x\| \le \max\left(\bigcup_{\{\ell : p_\ell > j\}} \{ \frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \|x_s^0\|_{N_{s-1}^0, p_\ell - j} : \mathrm{ran}\,(x_s^0) \subseteq F_\ell, s \ne m_\ell \} \cup \{ \frac{\theta_j}{\theta_1} \} \right) + \varepsilon.$$

Let $\mathrm{ran}\,(x_s^0) \subseteq F_\ell$ and $p_\ell > j$. If $s < m_\ell$ we have $j_{s+1} \le j_{m_\ell} \le p_\ell - j$ and so $\|x_s^0\|_{N_{s-1}^0, p_\ell - j} < \varepsilon$. If $s > m_\ell$ we have $j_s \ge j_{m_\ell+1} > p_\ell - j$ and so $\|x_s^0\|_{N_{s-1}^0, p_\ell - j} < \phi_{p_\ell - j}(1+\varepsilon)$. Note that

$$\frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \phi_{p_\ell - j} = \theta^j \phi_{p_\ell}$$

and therefore

$$\|x\| \le \theta^j \sup_{p \ge j} \phi_p (1+\varepsilon) \vee \frac{\theta_j}{\theta_1} + 2\varepsilon.$$

Note (by remark 4.6) that we can write $x = \sum_F a_i x_i$ for some set $F \in S_j$ where $a_i > 0$ for all $i \in F$ and $\sum_{i \in F} a_i = 1$. Therefore $\delta_j(Y) \le \|x\|$ and since $\varepsilon > 0$ is arbitrary we obtain the result. □

Note that theorem 4.4 does not necessarily give the best possible estimate for $\delta_j(Y)$. Indeed if $\theta_n = 2^{-n}$ for all n then $T = T(\theta_n, S_n)_{\mathbb{N}}$ and for all $Y \prec T$, $\delta_j(Y) = 2^{-j}$ [OTW]. Yet theorem 4.4 only gives $\delta_j(Y) \le 2^{-j+1}$. However we have the following estimate which does yield the proper estimate for Tsirelson's space.

Theorem 4.16 *Let $X = T(\theta_n, S_n)_{\mathbb{N}}$ be regular. Then for all $Y \prec X$ and $j \in \mathbb{N}$ we have*

$$\delta_j(Y) \le \theta^j \sup_{p \ge j} \frac{\phi_p}{\phi_{p-j}}.$$

Proof Let $Y \prec X$, $j \in \mathbb{N}$ and $\varepsilon > 0$. Since Y contains ℓ_1^n's uniformly, for all $N \in \mathbb{N}$ $\exists y \in Y$ with $1 = \|y\| \le \|y\|_{N,0} \le 1+\varepsilon$. (see eg [OTW] proposition 2.7). Therefore we may choose inductively a normalized block sequence (x_i) in Y so that for $i \in \mathbb{N}$ if $N_i = \max(\mathrm{ran}\,(x_i^0))$ w.r.t. (e_i) ($N_0 = 1$) then $\|x_i\|_{N_{i-1},0} \le 1+\varepsilon$. Note then that for every $i, p \in \mathbb{N}$, $\|x_i\|_{N_{i-1},p} \le \|x_i\|_{N_{i-1},0} \le 1+\varepsilon$. Apply corollary 4.14 (for an appropriate sequence (ε_i^k)) to obtain x a $(j, (\varepsilon_i^k), 1)$ average of (x_i) w.r.t. (e_i) with admissible averaging tree $(x_i^k)_{k=0,i=1}^{j,N^k}$ of (x_i) and maximum coordinates $(N_i^k)_{k=0,i=0}^{j,N^k}$ w.r.t. (e_i). Note then that for every $i, p \in \mathbb{N}$ we have that $\|x_i^0\|_{N_{i-1}^0,p} \le 1+\varepsilon$. By corollary 4.14 (2) there exist $m \in \mathbb{N}$, $F_1 < \cdots < F_m$ intervals in $\mathbb{N}$ which don't split the x_i^0's and integers $(p_\ell)_{\ell=1}^m$ with $p_\ell \ge j$ for all ℓ, such that

$$\|x\| \le \max\{ \frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \|x_i^0\|_{N_{i-1}^0, p_\ell - j} : \ell = 1, \ldots, m, \mathrm{ran}\,(x_i^0) \subseteq F_\ell \} + \varepsilon \le \theta^j \sup_{p \ge j} \frac{\phi_p}{\phi_{p-j}} (1+\varepsilon) + \varepsilon$$

and the result follows since $\varepsilon > 0$ is arbitrary. □

To estimate $\delta_j(Y)$ for $Y = X$ is easy as we see from the next

Theorem 4.17 *Let $X = T(\theta_n, S_n)_{n \in \mathbb{N}}$ be regular. Then for all $j \in \mathbb{N}$ we have $\delta_j(X) = \theta_j$.*

Proof Let $j \in \mathbb{N}$ and $\varepsilon > 0$. Apply corollary 4.14 for $(x_i) = (e_i)$, $M = j$, $N = 1$ and an appropriate sequence (ε_i^k), to obtain x, a $(j, (\varepsilon_i^k), 1)$ average of (e_i) w.r.t. (e_i) with admissible averaging tree $(x_i^k)_{k=0,i=1}^{j,N^k}$ and maximum coordinates $(N_i^k)_{k=0,i=0}^{j,N^k}$ w.r.t. (e_i). Then by (2) there exists $m \in \mathbb{N}$, $F_1 < \cdots < F_m$ intervals in $\mathbb{N}$ and integers $(p_\ell)_{\ell=1}^m$ with $p_\ell \geq j$ for all ℓ, such that

$$\|x\| \leq \max\{\frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \|x_i^0\|_{N_{i-1}^0, p_\ell - j} : \ell = 1, \ldots, m, \operatorname{ran}(x_i^0) \subseteq F_\ell\} + \varepsilon.$$

Since $(x_i^0)_{i=1}^{N^0}$ is a subsequence of (e_i), we have $\|x_i^0\|_{N_{i-1}^0, p_\ell - j} = \theta_{p_\ell - j}$ for every $i = 1, \ldots, N^0$ and $\ell = 1, \ldots, m$. Thus $\|x\| \leq \max_{1 \leq \ell \leq m} \theta_{p_\ell} + \varepsilon$. Since the sequence (θ_i) is decreasing we have $\|x\| \leq \theta_j + \varepsilon$. Since $\operatorname{supp}(x) \in S_j$ and $\varepsilon > 0$ is arbitrary we obtain the result. □

Question If $X = T(\theta_n, S_n)_{\mathbb{N}}$ is a regular mixed Tsirelson space and $Y \prec X$ is $\delta_j(Y) = \theta_j$ for every $j \in \mathbb{N}$?

References

[AA] D. Alspach, S. Argyros, *Complexity of weakly null sequences* , Dissertationes Mathematicae **321** (1992).

[AD] S. Argyros, I Deliyanni, *Examples of asymptotic ℓ_1 Banach spaces* , preprint.

[FJ] T. Fiegel, W. B. Johnson, *A uniformly convex Banach space which contains no ℓ_p* , Compositio Math. **29** (1974), 179-190.

[Ma] B. Maurey, *A remark about distortion*, Oper. Theory:Adv. Appl. **77** (1995), 131-142.

[MMT] B. Maurey, V. D. Milman, N. Tomczak-Jaegermann, *Asymptotic infinite dimensional theory of Banach spaces*, Operator Theory: Advances and Applications, **77** (1995), 149-175.

[MT] V. D. Milman, N. Tomczak-Jaegermann, *Asymptotic ℓ_p spaces and bounded distortions*, Contemp. Math. **144** (1993), 173-196.

[OS1] E. Odell, Th. Schlumprecht, *The distortion problem*, Acta Math. **173** (1994), 259-281.

[OS2] E. Odell, Th. Schlumprecht, *The distortion problem*, GAFA **3** (1993), 201-207.

[OS3] E. Odell, Th. Schlumprecht, *Distortion and stabilized structure in Banach spaces; New geometric phenomena for Banach and Hilbert spaces*, Proc. Inter. Cong. Math., Birkhäuser Verlag, Basel (1995), 955-965.

[OTW] E. Odell, N. Tomczak-Jaegermann, R. Wagner *Proximity to ℓ_1 and Distortion in asymptotic ℓ_1 spaces*, preprint.

[S] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81-95.

[T] N. Tomczak-Jaegermann, Banach spaces of type p have arbitrarily distortable subspaces, preprint.

[Ts] B. S. Tsirelson, *Not every Banach space contains ℓ_p or c_0*, Funct. Anal. Appl. **8** (1974), 138-141.

G. Androulakis, Dept of Math, RLM 8.100, The University of Texas at Austin, Austin, TX 78712
e-mail: giorgis@math.utexas.edu

E. Odell, Dept of Math, RLM 8.100, The University of Texas at Austin, Austin, TX 78712
e-mail: odell@math.utexas.edu