A note on the method of minimal vectors

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Abstract: The methods of "minimal vectors" were introduced by Ansari and Enflo and strengthened by Pearcy, in order to prove the existence of hyperinvariant subspaces for certain operators on Hilbert space. In this note we present the method of minimal vectors for operators on super-reflexive Banach spaces and we give a new sufficient condition for the existence of hyperinvariant subspaces of certain operators on these spaces..

1. Introduction

The *Invariant Subspace Problem (I.S.P.)* asks whether there exists a separable infinite dimensional Banach space on which every operator has a non-trivial invariant subspace. By "operator" we always mean "continuous linear map", by "subspace" we mean "closed linear manifold", and by "non-trivial" we mean "different than zero and the whole space". Several negative solutions to the I.S.P. are known [**4**] [**5**] [**13**] [**14**], [**15**], [**16**]. It remains unknown whether the separable Hilbert space is a positive solution to the I.S.P.. There is an extensive literature of results towards a positive solution of the I.S.P. especially in the case of the infinite dimensional separable complex Hilbert space ℓ_2. We only mention Lomonosov's result: every operator which is not a multiple of the identity and commutes with a non-zero compact operator on a complex Banach space has a non-trivial hyperinvariant subspace [**8**]. For surveys on the topic see [**12**] and [**9**]. Recently Ansari and Enflo [**1**] introduced the methods of minimal vectors and gave a new proof of the existence of non-trivial hyperinvariant subspaces of non-zero compact operators on ℓ_2. The method of minimal vectors which was introduced by Enflo, was strengthened by Pearcy [**10**] in order to give a new proof to the following special case of Lomonosov's theorem: every non-zero quasi-nilpotent operator on ℓ_2 which commutes with a non zero compact operator has a non-trivial hyperinvariant subspace. In this note we present the method of minimal vectors of an operator and we carry out two generalizations compared to the existing versions of Ansari-Enflo and Pearcy: Firstly, the operators are defined on a general super-reflexive Banach space rather than the space ℓ_2. This may be proved important if we try to find some Banach space which is a solution to the I.S.P. rather than examining whether ℓ_2 is a solution to the I.S.P.. Secondly, we introduce a property $(\star)$ that an operator may

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satisfy. If an operator Q commutes with a non-zero compact operator then Q satisfies property $(\star)$. Our main result (Theorem 2.2) refers to operators that satisfy property $(\star)$ rather than those that commute with a non-zero compact operator. More precisely, we prove that every non-zero quasi-nilpotent operator which satisfies property $(\star)$ on a super-reflexive Banach space has a non-trivial hyperinvariant subspace. We ask whether there exist operators which satisfy property $(\star)$ but do not have any non-zero compact operator in their commutant. Also we ask whether every operator with no non-trivial invariant subspace must satisfy property $(\star)$. If the answer is positive then Theorem 2.2 will imply that every quasi-nilpotent operator on a super-reflexive Banach space has a non-trivial invariant subspace. Then, every strictly singular operator on a super-reflexive Hereditarily Indecomposable complex Banach space has a non-trivial invariant subspace (see [**7**]), and hence the space constructed in [**6**] would provide a positive solution to the I.S.P..

We now recall some standard definitions and results that we shall use in this paper. A Banach space $(X, \|\cdot\|)$ is called *strictly convex* if for every $x, y \in X$ with $\|x\| = \|y\| = \|(x+y)/2\| = 1$ we have that $x = y$. A Banach space $(X, \|\cdot\|)$ is called *uniformly convex* if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\left\|\frac{x+y}{2}\right\| > 1 - \delta$ we have that $\|x - y\| < \varepsilon$. The function $\delta(\varepsilon)$ is called the modulus of uniform convexity of X. The norm of X is *Gâteaux differentiable* if for every $x \in X\backslash\{0\}$ and for every $y \in X$ the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists. The Banach space X is called *smooth* if for every $x \in X\backslash\{0\}$ there exists a unique $f \in X^*$ such that $f(x) = \|x\|^2 = \|f\|^2$. We denote the functional f by $(x)^*$. It can be proved that the norm of X is Gâteaux differentiable if and only if X is smooth, in which case the limit in (1) is equal to $\mathrm{Re}\,(x)^*(y)/\|(x)^*\|$. The norm of X is called *Fréchet differentiable* if the limit in (1) exists uniformly for all $y \in X$ with $\|y\| = 1$. The norm of X is called *uniformly smooth* if the limit in (1) exists uniformly for all $x, y \in X$ with $\|x\| = \|y\| = 1$. A Banach space X is called *super-reflexive* if every infinite dimensional space Y which is finitely represented in X must be reflexive. It is proved in [**3**] (see also [**11**]) that every super-reflexive Banach space X can be equivalently renormed to be uniformly convex. It follows from a renorming technique of Asplund [**2**] that a Banach space is super-reflexive if and only if it can be equivalently renormed to be uniformly convex or uniformly smooth or both.

2. Minimal vectors and invariant subspaces

We start by introducing some notations and terminology. If X is a Banach space, $x \in X$ and $\varepsilon > 0$ we denote by $\mathrm{S}(x, \varepsilon)$ (respectively $\mathrm{Ba}(x, \varepsilon)$) the *sphere* (respectively the *closed ball*) of X with center x and radius ε, namely the set $\{y \in X : \|x - y\| = \varepsilon\}$ (respectively the set $\{y \in X : \|x - y\| \leq \varepsilon\}$).

Definition 2.1. *Let X be a Banach space and Q be an operator on X. We say that an operator Q satisfies property $(\star)$ if for every $\varepsilon \in (0,1)$ there exists $x_0 \in S(0,1)$ such that for every weakly convergent sequence $(x_n) \subset S(x_0, \varepsilon)$ there exists a subsequence $(x_{n_k})_k$ of (x_n) and a sequence $(K_k) \subset \{Q\}'$ such that*

(a) $\|K_k\| \leq 1$ *and* $\|K_k(x_0)\| \geq \frac{1+\varepsilon}{2}$ *for all* $k \in \mathbb{N}$.
(b) $(K_k(x_{n_k}))_k$ *converges in norm.*

The purpose of (a) is to ensure that the limit of (b) is non-zero. Indeed, notice that if $Q, \varepsilon, x_0, (K_k)_k, (x_{n_k})_k$ are as in the previous definition then $\|K_k x_0 - K_k x_{n_k}\| \leq \|K_k\| \|x_0 - x_{n_k}\| \leq \varepsilon$, thus $\|K_k x_{n_k}\| \geq \|K_k x_0\| - \|K_k x_0 - K_k x_{n_k}\| \geq (1+\varepsilon)/2 - \varepsilon = (1-\varepsilon)/2 > 0$.

Also notice that for every operator Q if there exists a non-zero compact operator which commutes with Q then Q satisfies property $(\star)$.

Our main result is the following:

THEOREM 2.2. *Let X be a super-reflexive Banach space and Q be a non-zero quasi-nilpotent operator on X which satisfies property $(\star)$. Then Q has a non-trivial hyperinvariant subspace.*

For the proof of this result we use the method of minimal vectors of an operator. Let $(X, \|\cdot\|)$ be a reflexive Banach space, Q be an operator on X with dense range, $x_0 \in X$ with $\|x_0\| = 1$ and $\varepsilon \in (0,1)$. We define a sequence of *minimal vectors of Q with respect to x_0 and ε* to be a sequence $(y_n)_{n\in\mathbb{N}}$ as follows. For every $n \in \mathbb{N}$ the set $Q^{-n}\mathrm{Ba}(x_0, \varepsilon)$ is non-empty (since Q^n has a dense range), closed and convex. Thus there exists $y_n \in Q^{-n}\mathrm{Ba}(x_0,\varepsilon)$ such that

$$\|y_n\| = \inf\{\|y\| : y \in Q^{-n}\mathrm{Ba}(x_0, \varepsilon)\}. \tag{2}$$

Indeed, if $(y_{n,m})_m$ is a sequence in $Q^{-n}\mathrm{Ba}(x_0,\varepsilon)$ with

$$\|y_{n,m}\| \searrow \inf\{\|y\| : y \in Q^{-n}\mathrm{Ba}(x_0, \varepsilon)\}, \tag{3}$$

then $(y_{n,m})_m$ is a subset of $Q^{-n}\mathrm{Ba}(x_0,\varepsilon) \cap \mathrm{Ba}(0, \|y_{n,1}\|)$ which is weakly compact (since it is a closed, convex and bounded subset of a reflexive space). Thus by passing to a subsequence and relabeling we can assume that $(y_{n,m})_m$ converges weakly to some vector y_n. Since the norm is weakly lower semicontinuous, (3) implies (2).

In order to prove Theorem 2.2 we need the following three results whose proofs are postponed. For the first result, notice that if X is a reflexive smooth Banach space, Q is an operator on X with dense range, $x_0 \in X$ with $\|x_0\| = 1$, $\varepsilon \in (0,1)$ and (y_n) is a sequence of minimal vectors of Q with respect to x_0 and ε, then the sequence $((Q^n y_n - x_0)^*)_n$ is bounded, (namely, $\|(Q^n y_n - x_0)^*\| = \|Q^n y_n - x_0\| = \varepsilon$) by the minimality of $\|y_n\|$), thus it has weak* limit points. We want to know that 0 is not a weak* limit point of the sequence $((Q^n y_n - x_0)^*)_n$. The next result yields that this is true provided that the choice of ε is appropriate.

LEMMA 2.3. *Let $(X, \|\cdot\|)$ be a smooth and uniformly convex Banach space and Q be an operator on X with dense range. Then there exists $\varepsilon \in \left[\frac{1}{2}, 1\right)$ such that the following is satisfied: if $x_0 \in X$ with $\|x_0\| = 1$, $(y_n)_n$ is a sequence of minimal vectors of Q with respect to x_0 and ε, and f is a weak* limit point of $((Q^n y_n - x_0)^*)_n$, then $f \neq 0$.*

LEMMA 2.4. *Let X be a reflexive Banach space, Q be a quasi-nilpotent operator on X with dense range, $x_0 \in X$ with $\|x_0\| = 1$, $\varepsilon \in (0,1)$, and $(y_n)_{n\in\mathbb{N}}$ be a sequence of minimal vectors of Q with respect to x_0 and ε. Then there exists an increasing sequence $(n_k)_k$ of $\mathbb{N}$ such that*

$$\lim_k \frac{\|y_{n_k-1}\|}{\|y_{n_k}\|} = 0. \tag{4}$$

For the next Lemma, if X is a Banach space and $f \in X^*$ then $\ker(f)$ denotes the *kernel* of f.

LEMMA 2.5. *Let X be a reflexive smooth Banach space, Q be an operator on X with dense range, $x_0 \in X$ with $\|x_0\| = 1$, $\varepsilon \in (0,1)$, and $(y_n)_{n\in\mathbb{N}}$ be a sequence of minimal vectors of Q with respect to x_0 and ε. Then for all $n \in \mathbb{N}$,*

$$\ker((y_n)^*) \subseteq \ker((Q^n)^*(Q^n y_n - x_0)^*). \tag{5}$$

Now we are ready for the

PROOF OF THEOREM 2.2. Since X is super-reflexive we can assume by our discussion in the previous section, that $(X, \|\cdot\|)$ is smooth and uniformly convex. Without loss of generality we assume that Q has a dense range and it is 1-1 (because the range and the kernel of Q are hyperinvariant subspaces of Q). By Lemma 2.3 there exists $\varepsilon \in \left[\frac{1}{2}, 1\right)$ such that the conclusion of the lemma is satisfied. For that ε, since Q satisfies property $(\star)$, let $x_0 \in X$, $\|x_0\| = 1$ such that the statement of the definition of property $(\star)$ is valid for the operator Q. Let $(y_n)_n$ be a sequence of minimal vectors of Q with respect to x_0 and ε. By Lemma 2.4 let $(n_k)_k$ be an increasing subsequence of $\mathbb{N}$ such that (4) is valid. Since X is reflexive, by considering a further subsequence of (n_k) and relabeling we can assume that $(Q^{n_k-1}y_{n_k-1})_k$ converges weakly. By the property $(\star)$ of Q, there exists a subsequence of (n_k) (which, by relabeling, is still called (n_k)) and a sequence $(K_k)_k \subset \{Q\}'$ such that $(K_k Q^{n_k-1} y_{n_k-1})_k$ converges in norm to some vector $w \in X$. By our discussion following the definition of property $(\star)$ we have that $w \neq 0$. Since Q is 1-1 we have that $Qw \neq 0$. We claim that $Y := \{Q\}'(Qw)$ is a non-trivial hyperinvariant subspace for Q. We only need to show that $Y \neq X$. For that reason we let f to be a weak* limit point of $((Q^{n_k} y_{n_k} - x_0)^*)_k$, which is non-zero by Lemma 2.3, and we will show that $Y \subset \ker(f)$. We need to show that if $T \in \{Q\}'$ then $f(TQw) = 0$. Let $T \in \{Q\}'$ and $k \in \mathbb{N}$. Since $\ker((y_{n_k})^*)$ is a 1-codimensional subspace of X and $y_{n_k} \notin \ker((y_{n_k})^*)$ (notice that $(y_{n_k})^*(y_{n_k}) = \|y_{n_k}\|^2 \neq 0$), we have that $X = \operatorname{span}\{y_{n_k}\} \oplus \ker((y_{n_k})^*)$, thus there exists a scalar a_k and $r_k \in \ker((y_{n_k})^*)$ such that

$$TK_k(y_{n_k-1}) = a_k y_{n_k} + r_k. \tag{6}$$

We claim that $a_k \to 0$. Indeed,

$$\begin{aligned} |a_k| \|y_{n_k}\|^2 &= |(y_{n_k})^*(a_k y_{n_k} + r_k)| \\ &= |(y_{n_k})^* TK_k(y_{n_k-1})| \quad \text{(by (6))} \\ &\leq \|(y_{n_k})^*\| \|T\| \|K_k\| \|y_{n_k-1}\| \\ &\leq \|y_{n_k}\| \|T\| \|y_{n_k-1}\|, \end{aligned}$$

thus $|a_k| \leq \|T\| \|y_{n_k-1}\| / \|y_{n_k}\| \underset{k\to\infty}{\longrightarrow} 0$.

First apply Q^{n_k} and then $(Q^{n_k} y_{n_k} - x_0)^*$, on (6), to obtain

$$Q^{n_k} TK_k y_{n_k-1} = a_k Q^{n_k} y_{n_k} + Q^{n_k} r_k \quad \text{and}$$

$$(Q^{n_k} y_{n_k} - x_0)^* Q^{n_k} TK_k y_{n_k-1} = a_k (Q^{n_k} y_{n_k} - x_0)^* Q^{n_k} y_{n_k} + (Q^{n_k} y_{n_k} - x_0)^* Q^{n_k} r_k. \tag{7}$$

Since $r_k \in \ker((y_{n_k})^*)$, by Lemma 2.5 we have that $(Q^{n_k} y_{n_k} - x_0)^* Q^{n_k}(r_k) = 0$. Furthermore, since $a_k \to 0$ and $\|Q^{n_k} y_{n_k} - x_0\| = \varepsilon$, we have that the right hand side of (7) tends to zero. Thus by taking limits and noticing that $T, K_k \in \{Q\}'$ for all k, (7) becomes

$$\lim_k (Q^{n_k} y_{n_k} - x_0)^* TQK_k Q^{n_k-1} y_{n_k-1} = 0. \tag{8}$$

Since $(K_k Q^{n_k-1} y_{n_k-1})_k$ converges in norm to w and f is a weak* limit point of $((Q^{n_k} y_{n_k} - x_0)^*)_k$, (8) yields that $f(TQw) = 0$ which finishes the proof. $\square$

We now turn our attention to the proof of Lemma 2.3. Before presenting the proof of Lemma 2.3 we need the following two results.

SUBLEMMA 2.6. *Let $(X, \|\cdot\|)$ be a smooth and strictly convex Banach space, $x_0 \in X$, $\|x_0\| = 1$, $0 < \varepsilon < 1$ and $w \in S(x_0, \varepsilon)$. The following conditions are equivalent:*

(a) $\|x_0 - \lambda w\| > \varepsilon$ *for all* $\lambda \in [0, 1)$.

(b) *Re* $\frac{(x_0-w)^*}{\|x_0-w\|^2}(x_0) \geq 1$.

PROOF. (a) $\Rightarrow$ (b): Since $\|x_0 - \lambda w\| > \varepsilon = \|x_0 - 1 \cdot w\|$, for all $\lambda \in [0, 1)$, we have that the derivative of the function $f(\lambda) = \|x_0 - \lambda w\|$ at 1 is non-positive. Set $g(\mu) = f(1 - \mu)$. Then $g'(0) = -f'(1)$, thus $g'(0) \geq 0$. Note that $g(\mu) = \|x_0 - (1 - \mu)w\| = \|x_0 - w + \mu w\|$. Thus

$$\begin{aligned} g'(0) &= \operatorname{Re} \frac{(x_0 - w)^*}{\|(x_0 - w)^*\|}(w) = \operatorname{Re} \frac{(x_0 - w)^*}{\|x_0 - w\|}(w) \\ &= \operatorname{Re} \frac{(x_0 - w)^*}{\|x_0 - w\|}(w - x_0 + x_0) = -\frac{\|x_0 - w\|^2}{\|x_0 - w\|} + \operatorname{Re} \frac{(x_0 - w)^*}{\|x_0 - w\|}(x_0) \\ &= -\|x_0 - w\| + \operatorname{Re} \frac{(x_0 - w)^*}{\|x_0 - w\|}(x_0). \end{aligned}$$

Thus $g'(0) \geq 0$ if and only if

$$\operatorname{Re} \frac{(x_0 - w)^*}{\|x_0 - w\|^2}(x_0) \geq 1.$$

(b) $\Rightarrow$ (a): By the proof of (a) $\Rightarrow$ (b), notice that if (b) is valid then $f'(1) \leq 0$. Notice also that f is a strictly convex function (since $(X, \|\cdot\|)$ is strictly convex) with $f(1) = \varepsilon$ (since $w \in \mathrm{S}(x_0, \varepsilon)$). Thus if (b) is valid then $f(\lambda) > \varepsilon$ for $\lambda \in [0, 1)$. $\square$

LEMMA 2.7. *If $(X, \|\cdot\|)$ is a smooth and uniformly convex Banach space, then for every $\eta > 0$ there exists $\varepsilon \in \left[\frac{1}{2}, 1\right)$ such that for all $x_0 \in X$ with $\|x_0\| = 1$ and for all $w \in S(x_0, \varepsilon)$ satisfying $\|x_0 - \lambda w\| > \varepsilon$ for all $\lambda \in [0, 1)$, we have that $\|w\| \leq \eta$.*

PROOF. We start with the

Claim: Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, and $\eta' > 0$. Let $\delta(\cdot)$ denote the modulus of uniform convexity of X. Then for every $x_0 \in X$ with $\|x_0\| = 1$,

$$\sup\{\|x\| \colon\ x \in \mathrm{S}(x_0, 1),\ \operatorname{Re}(x_0 - x)^*(x_0) > 1 - 2\delta(\eta')\} \leq \eta'. \tag{9}$$

Indeed, if $x \in \mathrm{S}(x_0, 1)$ with $\|x\| \geq \eta'$, then we have that $\|x_0 - (x_0 - x)\| \geq \eta'$, $\|x_0\| = 1$, $\|x_0 - x\| = 1$, hence

$$\begin{aligned} 1 - \delta(\eta') &\geq \left\| \frac{x_0 + (x_0 - x)}{2} \right\| \geq \left| (x_0 - x)^* \left(\frac{x_0 + (x_0 - x)}{2} \right) \right| \\ &\geq \operatorname{Re}(x_0 - x)^* \left(\frac{x_0 + (x_0 - x)}{2} \right) \\ &= \frac{\operatorname{Re}(x_0 - x)^*(x_0) + 1}{2}. \end{aligned}$$

Thus $\operatorname{Re}(x_0-x)^*(x_0) \le 1-2\delta(\eta')$ which finishes the proof of the Claim.

Let X and η as in the statement of Lemma 2.7. Let ε satisfying

$$\frac{1}{2} \le \varepsilon, \quad 0 < \frac{1}{\varepsilon} - 1 \le 2\delta\left(\frac{\eta}{2}\right) \quad \text{and} \quad 1-\varepsilon \le \eta/2. \tag{10}$$

Let $w \in \mathrm{S}(x_0, \varepsilon)$ satisfying $\|x_0 - \lambda w\| > \varepsilon$ for all $\lambda \in [0,1)$. By Sublemma 2.6 we have that

$$\operatorname{Re} \frac{(x_0-w)^*}{\|x_0-w\|^2}(x_0) \ge 1. \tag{11}$$

Let $x \in \mathrm{S}(x_0, 1)$ with

$$x_0 - w = \varepsilon(x_0 - x). \tag{12}$$

Then

$$\begin{aligned}
\left|\operatorname{Re} \frac{(x_0-w)^*}{\|x_0-w\|^2}(x_0) - \operatorname{Re}(x_0-x)^*(x_0)\right| &\le \left|\frac{(x_0-w)^*}{\|x_0-w\|^2}(x_0) - (x_0-x)^*(x_0)\right| \\
&\le \|x_0\| \left\|\frac{(x_0-w)^*}{\|x_0-w\|^2} - (x_0-x)^*\right\| \\
&= \left\|\frac{\varepsilon(x_0-x)^*}{\|\varepsilon(x_0-x)\|^2} - (x_0-x)^*\right\| \quad \text{(by (12))} \\
&\le \left|\frac{1}{\varepsilon} - 1\right| \le 2\delta\left(\frac{\eta}{2}\right) \quad \text{(by (10))}.
\end{aligned}$$

Therefore by (11) we have that $\operatorname{Re}(x_0-x)^*(x_0) \ge 1 - 2\delta\left(\frac{\eta}{2}\right)$. By (9) we have that $\|x\| \le \eta/2$. By (12) we have that $w = (1-\varepsilon)x_0 + \varepsilon x$ and thus by the triangle inequality, $\|x\| \le \eta/2$, and (10) we obtain $\|w\| \le (1-\varepsilon) + \varepsilon\|x\| \le (1-\varepsilon) + \|x\| \le \eta$. □

Proof of Lemma 2.3. Let X be a smooth and uniformly convex Banach space and Q be an operator on X with dense range. For $\eta = \frac{1}{3}$ we choose $\varepsilon \in \left[\frac{1}{2}, 1\right)$ to satisfy the statement of Lemma 2.7.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of minimal vectors of Q. For $n \in \mathbb{N}$, by the minimality of $\|y_n\|$ we have that $Q^n y_n \in \mathrm{S}(x_0, \varepsilon)$ and $\|x_0 - \lambda Q^n y_n\| > \varepsilon$ for all $\lambda \in [0,1)$. Thus by Lemma 2.7 we obtain $\|Q^n y_n\| \le \frac{1}{3}$. Let f be any weak* limit point of the sequence $((Q^n y_n - x_0)^*)_{n\in\mathbb{N}}$. Since $\|x_0\| = 1$, $\|Q^n y_n\| \le \frac{1}{3}$ and $\|Q^n y_n - x_0\| = \varepsilon \ge 1/2$ we have that $f \ne 0$. Indeed for $n \in \mathbb{N}$,

$$\begin{aligned}
|(Q^n y_n - x_0)^*(-x_0)| &= |(Q^n y_n - x_0)^*(Q^n y_n - x_0 - Q^n y_n)| \\
&= |(Q^n y_n - x_0)^*(Q^n y_n - x_0) - (Q^n y_n - x_0)^*(Q^n y_n)| \\
&= |\|Q^n y_n - x_0\|^2 - (Q^n y_n - x_0)^*(Q^n y_n)| \\
&\ge \varepsilon^2 - |(Q^n y_n - x_0)^*(Q^n y_n)| \\
&\ge \varepsilon^2 - \|(Q^n y_n - x_0)^*\|\|Q^n y_n\| \\
&\ge \varepsilon^2 - \varepsilon \cdot \frac{1}{3} \\
&\ge \frac{1}{4} - \frac{1}{6} = \frac{1}{12} > 0 \quad \left(\text{since } \frac{1}{2} \le \varepsilon\right).
\end{aligned}$$

Since f is a weak* limit point of $((Q^n y_n - x_0)^*_n$, we have that $|f(-x_0)| \ge \frac{1}{12}$, thus $f \ne 0$. □

PROOF OF LEMMA 2.4. If the statement is not true, then there exists a positive number δ such that

$$\frac{\|y_{n-1}\|}{\|y_n\|} \geq \delta \text{ for all } n \in \mathbb{N}.$$

Thus for every integer $n \in \mathbb{N}$, we have

$$\|y_1\| \geq \delta\|y_2\| \geq \delta^2\|y_3\| \geq \cdots \geq \delta^n\|y_{n+1}\|. \tag{13}$$

We have that $\|Qy_1 - x_0\| \leq \varepsilon$. Also we have that $\|Q(Q^n y_{n+1}) - x_0\| = \|Q^{n+1}y_{n+1} - x_0\| \leq \varepsilon$. By the minimality of $\|y_1\|$ we have:

$$\|y_1\| \leq \|Q^n y_{n+1}\| \leq \|Q^n\|\|y_{n+1}\|. \tag{14}$$

By combining (13) and (14) we have $\delta \leq \|Q^n\|^{1/n}$ which is a contradiction since Q is quasi-nilpotent. □

In order to prove Lemma 2.5 we need the following

REMARK 2.8. *Let X be a Banach space, $f \in X^*\backslash\{0\}$ and $g \in X^*$ such that*

$$\text{for all } x \in X, \text{ if } \operatorname{Re}(f(x)) < 0 \text{ then } \operatorname{Re}(g(x)) \geq 0. \tag{15}$$

Then $\ker(f) \subseteq \ker(g)$ and moreover there exists a non-positive real number a such that $g = af$.

PROOF. Let $x_0 \in X$ with $f(x_0) = -1$. Thus $\operatorname{Re} g(x_0) \geq 0$. We first claim that $\ker(f) \subseteq \ker(g)$. Indeed, otherwise there exists $x \in \ker(f)\backslash\ker(g)$. Without loss of generality assume that $\operatorname{Re} g(x) < -2\operatorname{Re} g(x_0)$. Let $x' = x_0 + x$. Then $f(x') = f(x_0) + f(x) = f(x_0)$, hence $\operatorname{Re} f(x') = -1 < 0$. We also have $\operatorname{Re} g(x') = \operatorname{Re} g(x_0) + \operatorname{Re} g(x) < -\operatorname{Re} g(x_0) \leq 0$ which is a contradiction, proving that $\ker(f) \subseteq \ker(g)$.

Since both $\ker(f), \ker(g)$ are at most 1-codimensional subspaces of X we have that there exists a scalar a such that $g = af$. Since $f(x_0) = -1$ and $\operatorname{Re} g(x_0) \geq 0$ we have that $\operatorname{Re} a \leq 0$. If $a \notin \mathbb{R}$ then let $a = a_1 + ia_2$ with $a_1, a_2 \in \mathbb{R}$, $a_1 \leq 0$ and $a_2 \neq 0$. Let $x_1 \in X$ with $f(x_1) = -1 + ia_2^{-1}(1 - a_1)$. Then $\operatorname{Re} f(x_1) = -1 < 0$ and $\operatorname{Re} g(x_1) = \operatorname{Re}((a_1 + ia_2)(-1 + ia_2^{-1}(1 - a_1))) = -1 < 0$ which is a contradiction, proving that $a \in \mathbb{R}$. □

Now we are ready for the

PROOF OF LEMMA 2.5. For a fixed $n \in \mathbb{N}$ we prove that the assumption of Remark 2.8 is satisfied for $f = (y_n)^*$ and $g = (Q^n)^*((Q^n y_n - x_0)^*)$. Let $x \in X$ with $\operatorname{Re}(y_n)^*(x) < 0$. We claim that $\operatorname{Re}(Q^n)^*(Q^n y_n - x_0)^*(x) \geq 0$ i.e. $\operatorname{Re}(Q^n y_n - x_0)^*(Q^n x) \geq 0$. Indeed, otherwise, since $\operatorname{Re}(Q^n y_n - x_0)^*(Q^n x)/\|(Q^n y_n - x_0)^*\|$ is the derivative of the function

$$t \mapsto \|Q^n y_n - x_0 + tQ^n x\|$$

at 0, we obtain that this function is decreasing for t in a neighborhood of 0. Thus for small $t > 0$ we have

$$\varepsilon = \|Q^n y_n - x_0\| \geq \|Q^n y_n - x_0 + tQ^n x\|$$

i.e.

$$\|Q^n(y_n + tx) - x_0\| \leq \varepsilon.$$

We have by the minimality of $\|y_n\|$ that

$$\|y_n\| \leq \|y_n + tx\| \quad \text{for small} \quad t > 0.$$

Thus the derivative of the function

$$t \mapsto \|y_n + tx\|$$

must be non-negative at 0, i.e. $\operatorname{Re}(y_n)^*(x) \geq 0$ which is a contradiction. □

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