A COUNTEREXAMPLE TO A QUESTION OF R. HAYDON, E. ODELL AND H. ROSENTHAL

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Abstract: We give an example of a compact metric space K, an open dense subset U of K, and a sequence (f_n) in $C(K)$ which is pointwise convergent to a non-continuous function on K, such that for every $u \in U$ there exists $n \in \mathbf{N}$ with $f_n(u) = f_m(u)$ for all $m \geq n$, yet (f_n) is equivalent to the unit vector basis of the James quasi-reflexive space of order 1. Thus c_0 does not embed isomorphically in the closed linear span $[f_n]$ of (f_n). This answers in negative a question asked by H. Haydon, E. Odell and H. Rosenthal.

1. Introduction

A result of J. Elton [E], which was also proved later by R. Haydon, E. Odell and H. Rosenthal [HOR], states that if K is a compact metric space, and (f_n) is a uniformly bounded sequence in $C(K)$ such that

$$\sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty, \ \forall k \in K$$

and the pointwise limit of (f_n) on K is a non-continuous function, then c_0 embeds isomorphically in the closed linear span $[f_n]$ of (f_n). Thus the following question was naturally raised by R. Haydon, E. Odell and H. Rosenthal:

Question 4.7 in [HOR]**:** Let K be a compact metric space, R be a residual subset of K (i.e. $K \backslash R$ is a first category set), and (f_n) be a sequence in $C(K)$ which converges pointwise on K to a non-continuous function, and

$$\sum_{n=1}^{\infty} |f_{n+1}(r) - f_n(r)| < \infty, \text{ for all } r \in R.$$

Does c_0 embed in the closed linear span $[f_n]$ of (f_n)?

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We will construct K a compact metric space, U an open dense subset of K and a sequence $(g_n) \subset C(K)$ such that

(a) $(\sum_{i=1}^{n} g_i)_n$ is a uniformly bounded and pointwise convergent sequence on K to a non-continuous function;
(b) For every $u \in U$ there exists $n \in \mathbf{N}$ such that $g_m(u) = 0$ for every $m \geq n$;
(c) $[g_n]$ is isomorphic to the James quasi-reflexive of order 1 space J.

Since, of course, c_0 does not embed isomorphically in J, this answers in the negative Question 4.7 of [HOR]. Our construction is very elementary and explicit even though a shorter proof of the existence of a counterexample to Question 4.7 of [HOR] can be given along similar lines using more advanced machinery.

2. The construction

We recall the definition of the James space J and some simple facts. Let c_{00} denote the finitely supported sequences of real numbers. For $(x_n) \in c_{00}$ we define

$$\|(x_n)\|_J = \sup\{[x_{p_1}^2 + (x_{p_2} - x_{p_1})^2 + \cdots + (x_{p_k} - x_{p_{k-1}})^2]^{1/2} : \\ k \in \mathbf{N}, 1 \leq \mathrm{p}_1 < \mathrm{p}_2 < \cdots < \mathrm{p}_{k-1} < \mathrm{p}_k\}.$$

Then the James space J is the completion of $(c_{00}, \|\cdot\|_J)$. If (e_n) is the unit vector basis of c_{00}, then (e_n) becomes the unit vector basis of J, which is monotone and shrinking. Also, $(\sum_{i=1}^{n} e_n)_n$ is a weak-Cauchy sequence which is not weakly convergent in J. If $(a_n) \in c_0$ such that (a_n) is a monotone sequence of real numbers (i.e. non-increasing, or non-decreasing) then $\|(a_n)\|_J = |a_1|$ (this is because if $a, b \in \mathbf{R}$ with $ab \geq 0$, then $a^2 + b^2 \leq (a+b)^2$).

Notation: For $(a_n), (b_n) \in c_{00}$, we define $(a_n) \cdot (b_n) \in c_{00}$, by

$$(a_n) \cdot (b_n) = (a_n b_n).$$

Lemma 2.1. *For $(a_n), (b_n) \in c_{00}$ we have*

$$\|(a_n) \cdot (b_n)\|_J \leq \|(a_n)\|_J \|(b_n)\|_\infty + \|(a_n)\|_\infty \|(b_n)\|_J.$$

Proof For some $k \in \mathbf{N}$ and some finite sequence of positive integers $1 \leq p_1 < p_2 < \cdots p_k$ we have:

$$\begin{aligned} \|(a_n) \cdot (b_n)\|_J &= [(a_{p_1} b_{p_1})^2 + (a_{p_2} b_{p_2} - a_{p_1} b_{p_1})^2 + \cdots + (a_{p_k} b_{p_k} - a_{p_{k-1}} b_{p_{k-1}})^2]^{1/2} \\ &= [(a_{p_1} b_{p_1})^2 + (a_{p_2}(b_{p_2} - b_{p_1}) + (a_{p_2} - a_{p_1}) b_{p_1})^2 + \cdots + \\ &\quad (a_{p_k}(b_{p_k} - b_{p_{k-1}}) + (a_{p_k} - a_{p_{k-1}}) b_{p_{k-1}})^2]^{1/2}. \end{aligned}$$

Therefore by the triangle inequality in ℓ_2 we have that

$$\begin{aligned}\|(a_n)\cdot(b_n)\|_J &\le [a_{p_1}^2 b_{p_1}^2 + (a_{p_2}-a_{p_1})^2 b_{p_1}^2 + \cdots + (a_{p_k}-a_{p_{k-1}})^2 b_{p_{k-1}}^2]^{1/2} + \\ &\quad [a_{p_2}^2 (b_{p_2}-b_{p_1})^2 + \cdots + a_{p_k}^2 (b_{p_k}-b_{p_{k-1}})^2]^{1/2} \\ &\le [a_{p_1}^2 + (a_{p_2}-a_{p_1})^2 + \cdots + (a_{p_k}-a_{p_{k-1}})^2]^{1/2} \|(b_n)\|_\infty + \\ &\quad \|(a_n)\|_\infty [(b_{p_2}-b_{p_1})^2 + \cdots + (b_{p_k}-b_{p_{k-1}})^2]^{1/2} \\ &\le \|(a_n)\|_J \|(b_n)\|_\infty + \|(a_n)\|_\infty \|(b_n)\|_J\end{aligned}$$

which finishes the proof of the lemma. □

Now we are ready to see the counterexample. Let $K := \{(a,b) \in \mathbf{R}^2 : 0 \le \mathrm{a} \le 1, 0 \le \mathrm{b} \le 1\}$. Since $C[0,1]$ is universal for the class of separable spaces, there exists a sequence $(f_n) \subset C[0,1]$, and $M > 0$ such that (f_n) is M-equivalent to the unit vector basis of J. For $n \in \mathbf{N}$ set $K_n := \{(a,b) \in \mathbf{R}^2 : 0 \le \mathrm{a} \le 1, 1/2^{\mathrm{n}} \le \mathrm{b} \le 1\}$, $L_n := \{(a,b) \in \mathbf{R}^2 : 0 \le \mathrm{a} \le 1, \mathrm{b} = 1/2^{\mathrm{n}}\}$ $L := \{(a,0) : 0 \le a \le 1\}$ and $U = K \backslash L$. Now, for $n \in \mathbf{N}$ define $g_n : K \to \mathbf{R}$ by

- $g_n \mid K_n \equiv 0$,
- for every $0 \le a \le 1$, g_n restricted to the segment connecting the points $(a, 1/2^n)$ and $(a,0)$, is linear,
- $g_n \mid L \equiv f_n$.
- g_n is continuous on K.

We will show that (g_n) is equivalent to the unit vector basis (e_i) of the James space. This will imply that $(\sum_{i=1}^n g_i)_n$ is a weak Cauchy sequence which is not weakly convergent, which will finish the proof. Let $n \in \mathbf{N}$ and $(\lambda_i)_{i=1}^n \subset \mathbf{R}$. We want to estimate $\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty$. For $(a,b), (c,d) \in K$, let $[(a,b),(c,d)]$ denote the linear segment connecting the points (a,b) and (c,d). For every $0 \le a \le 1$ we have that

- $(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid [(a,1),(a,1/2)] \equiv 0$,
- $(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid [(a,1/2^i),(a,1/2^{i+1})]$ is linear, for every $i = 1, \ldots, n-1$,
- $(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid [(a,1/2^n),(a,0)]$ is linear,
- $\lambda_1 g_1 + \cdots + \lambda_n g_n$ is continuous on K.

Therefore we obtain:

$$\begin{aligned}&\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty \\ &\qquad = \max_{2\le k\le n} \|(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid L_k\|_\infty \vee \|(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid L\|_\infty \\ &\qquad = \max_{2\le k\le n} \|(\lambda_1 g_1 + \cdots \lambda_{k-1} g_{k-1}) \mid L_k\|_\infty \vee \|\lambda_1 f_1 + \cdots + \lambda_n f_n\|_\infty.\end{aligned}$$

Therefore we obtain immediately the lower estimate:

$$\begin{aligned}\|\lambda_1 g_1+\cdots+\lambda_n g_n\|_\infty &\geq \|\lambda_1 f_1+\cdots+\lambda_n f_n\|_\infty\\ &\geq \frac{1}{M}\|\lambda_1 e_1+\cdots+\lambda_n e_n\|_J.\end{aligned}$$

For the upper estimate we need to estimate $\|(\lambda_1 g_1+\cdots+\lambda_n g_n \mid L_k)\|_\infty$ for $2\leq k\leq n$. Note that for $0\leq a\leq 1$ and $2\leq k\leq n$ we have that

$$\begin{aligned}&(\lambda_1 g_1+\cdots+\lambda_n g_n)(a,1/2^k)\\ &\qquad=\lambda_1\frac{\frac{1}{2}-\frac{1}{2^k}}{\frac{1}{2}}f_1(a)+\lambda_2\frac{\frac{1}{2^2}-\frac{1}{2^k}}{\frac{1}{2^2}}f_2(a)+\cdots+\lambda_{k-1}\frac{\frac{1}{2^{k-1}}-\frac{1}{2^k}}{\frac{1}{2^{k-1}}}f_{k-1}(a)\\ &\qquad=\lambda_1\frac{2^{k-1}-1}{2^{k-1}}f_1(a)+\lambda_2\frac{2^{k-2}-1}{2^{k-2}}f_2(a)+\cdots+\lambda_{k-1}\frac{2-1}{2}f_{k-1}(a).\end{aligned}$$

Therefore we have that

$$\begin{aligned}&\|\lambda_1 g_1+\cdots+\lambda_{k-1}g_{k-1}\mid L_k\|_\infty\\ &\qquad=\|\lambda_1\frac{2^{k-1}-1}{2^{k-1}}f_1+\lambda_2\frac{2^{k-2}-1}{2^{k-2}}f_2+\cdots+\lambda_{k-1}\frac{2-1}{2}f_{k-1}\|_\infty\\ &\qquad\leq M\|\lambda_1\frac{2^{k-1}-1}{2^{k-1}}e_1+\lambda_2\frac{2^{k-2}-1}{2^{k-2}}e_2+\cdots+\lambda_{k-1}\frac{2-1}{2}e_{k-1}\|_J\\ &\qquad=M\|(\lambda_1,\lambda_2,\cdots,\lambda_{k-1},0,\dots)\\ &\qquad\qquad\cdot(\frac{2^{k-1}-1}{2^{k-1}},\frac{2^{k-2}-1}{2^{k-2}},\dots,\frac{2-1}{2},0,\dots)\|_J\\ &\qquad\leq M\|\lambda_1 e_1+\cdots+\lambda_{k-1}e_{k-1}\|_J\cdot 1\\ &\qquad+M\|(\lambda_i)_{i=1}^{k-1}\|_\infty\|(\frac{2^{k-1}-1}{2^{k-1}},\dots,\frac{2-1}{2},0,\dots)\|_J(\text{by Lemma 2.1})\\ &\qquad\leq M\|\lambda_1 e_1+\cdots+\lambda_{k-1}e_{k-1}\|_J+M\|(\lambda_i)\|_\infty\frac{2^{k-1}-1}{2^{k-1}}(\text{since the}\\ &\qquad\text{sequence }(\frac{2^{k-1}-1}{2^{k-1}},\frac{2^{k-2}-1}{2^{k-2}},\dots,\frac{2-1}{2},0,\dots,)\text{ is decreasing})\\ &\qquad\leq 2M\|\lambda_1 e_1+\cdots+\lambda_{k-1}e_{k-1}\|_J(\text{since }\|(\lambda_i)_{i=1}^{k-1}\|_\infty\leq\|(\lambda_i)_{i=1}^{k-1}\|_J).\end{aligned}$$

Also, since $\|\lambda_1 f_1+\cdots+\lambda_n f_n\|_J\leq M\|\lambda_1 e_1+\cdots+\lambda_n e_n\|_J$, we obtain that

$$\|\lambda_1 g_1+\cdots+\lambda_n g_n\|_\infty\leq 2M\|\lambda_1 e_1+\cdots+\lambda_n e_n\|_J.$$

This finishes the proof. □

References

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