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A new method for constructing invariant subspaces

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Abstract

The new idea that is used in this article for producing non-trivial (closed) invariant subspaces of (bounded linear) operators on reflexive Banach spaces, is the use of fixed points of set-valued functions. The advantage of this new method is that it is reasonable to expect that the famous method of Lomonosov for producing invariant subspaces using fixed points of functions, can be viewed as a special case of the use of fixed points of set-valued functions. Further uses of this new idea and open questions are suggested at the end of the article.

Keywords: Invariant subspaces; Fixed points of set-valued functions

1. Introduction

Let X be a Banach space (always of dimension larger than 1) and $T \in \mathcal{L}(X)$ (i.e. $T : X \to X$ is a (bounded linear) operator). A (closed) subspace Y of X is called an invariant subspace of T if $T(Y) \subseteq Y$. Also Y is called non-trivial if $\{0\} \subsetneqq Y \subsetneqq X$. The invariant subspace problem asks whether every operator on a complex separable Hilbert space has a non-trivial invariant subspace. This problem has its origins approximately in 1935 when (according to [6]) J. von Neumann proved (unpublished) that every compact operator on a separable infinite dimensional complex Hilbert space has a non-trivial invariant subspace (the proof uses the spectral theorem for normal operators, see [43]). Since then the invariant subspace problem has motivated enormous literature in operator theory. The books [10,12,32,36], the lecture notes [1] and [22], and the survey papers [20] and [2] are centered around the invariant subspace problem. Related open problems and

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conjectures appear in [3]. The invariant subspaces appear in a natural way in prediction theory (see A.N. Kolmogorov [25], and N. Wiener [44]), and in mathematical physics.

Let us recall some basic terminology and elementary facts about invariant subspaces: let X be a Banach space and $T \in \mathcal{L}(X)$. The operator T is called *non-transitive* if it has a non-trivial invariant subspace. For $x \in X \setminus \{0\}$ let $\mathrm{Orb}_T(x) = \{T^n x\colon n \in \mathbb{N} \cup \{0\}\}$ be the orbit of x under T. The closed linear space generated by $\mathrm{Orb}_T(x)$, $\overline{\mathrm{Span}}\,\mathrm{Orb}_T(x)$, is an invariant subspace of T. The vector $x \in X \setminus \{0\}$ is called *non-cyclic* for T if $\overline{\mathrm{Span}}\,\mathrm{Orb}_T(x) \neq X$. If the operator T is non-transitive then it has a non-cyclic vector. Also if X is a non-separable Banach space and T is an operator on X then every non-zero vector in X is non-cyclic for T, thus T is non-transitive. If X is a finite dimensional (always of dimension larger than 1) complex or odd dimensional real Banach space then the Fundamental Theorem of Algebra gives that T has a non-zero eigenvector which spans a non-trivial invariant subspace. If X is an even dimensional real Banach space with dimension larger than 2 then by considering its complexification it can be proved that every operator $T \in \mathcal{L}(X)$ has a non-trivial invariant subspace. On $\mathbb{R}^2$ the rotation by $\pi/2$ radians does not have non-trivial invariant subspaces. If $T \neq 0$ and $Y := \ker(T) \neq \{0\}$ then Y is a non-trivial (hyper-)invariant subspace of T (i.e. $SY \subseteq Y$ for all $S \in \{T\}' = \{A \in \mathcal{L}(X)\colon\ AT = TA\}$, the commutant of T). If $T \neq 0$ and $Y := \overline{\mathrm{Ran}(T)} \neq X$ then Y is a non-trivial hyper-invariant subspace of this X.

The remainder of this section contains some (non-exhaustive) history. Many important directions will not be discussed here. For some of these the reader is referred to [1,2,10,12,20,22, 32,36]. The history here is organized into five subsections: (A) the theorem of Aronszajn–Smith and some extensions; (B) the theorem of Lomonosov and some extensions; (C) subnormal, hyponormal operators and dual algebras; (D) the method of extremal vectors; (E) applications of the smooth variational principle and (F) examples of transitive operators.

(A) In 1954, N. Aronszajn and K.T. Smith [6] proved that if X is an infinite dimensional complex Banach space and $T \in \mathcal{L}(X)$ is completely continuous then T has a non-trivial invariant subspace. A non-linear map is used in the proof: X is assumed without loss of generality to be strictly convex and for a finite dimensional subspace Y of X, the "metric projection" $P_Y : X \to Y$ (a non-necessarily linear map) is defined by $P_Y(x)$ to be the unique $y \in Y$ which minimizes $\|x - y'\|$ for $y' \in Y$.

In 1966, A.R. Bernstein and A. Robinson [13] proved that if H is a complex Hilbert space, $T \in \mathcal{L}(H)$ is a polynomially compact operator (i.e. for some non-zero polynomial p, $p(T)$ is compact) then T has a non-trivial invariant subspace. The proof uses non-standard analysis as well as techniques similar to [6]. In 1966, P.R. Halmos gave a proof of the same result by a similar method but avoiding the non-standard tools, [21].

In 1968, W.B. Arveson and J. Feldman [7] proved the following: let H be a Hilbert space, and $T \in \mathcal{L}(H)$ satisfy $\|TP_n - P_n T P_n\| \to 0$ for some sequence (P_n) of orthogonal projection operators which converges strongly to the identity operator (such operators are called quasitriangular; the terminology is due to Halmos). Assume also that the norm closed algebra generated by T and 1 contains a non-zero compact operator. Then T has a non-trivial invariant subspace. In 1973, C. Pearcy and N. Salinas [33] proved that if T is a quasitriangular operator on a Hilbert space and $\mathcal{R}(T)$—the norm closure of the rational functions of T—contains a non-zero compact operator then there exists a non-trivial subspace invariant under all operators in $\mathcal{R}(T)$. If X is a Banach space and $\mathcal{A}$ is an algebra in $\mathcal{L}(X)$ then $\mathcal{A}$ is called non-transitive if there exists a non-trivial subspace which is invariant under every element of $\mathcal{A}$. (If $\mathcal{A} = \{T\}'$ then $\mathcal{A}$ is non-transitive if and only if T has a non-trivial hyper-invariant subspace.)

(B) In 1973, V.I. Lomonosov [26] proved the following celebrated result: let X be a complex Banach space and T be an operator on X which is not a multiple of the identity and commutes with some non-zero compact operator. Then T has a non-trivial hyper-invariant subspace. For the proof of this result, the new idea that was introduced was the Schauder fixed point theorem: if Φ is a norm-continuous function defined on a closed convex subset C of a normed space, and $\Phi(C)$ is contained in a norm-compact subset K of C, then Φ has a fixed point. Lomonosov's result was extended to real Banach spaces by N.D. Hooker in 1981, [23]. A special case of Lomonosov's theorem with a short proof (still using the Schauder fixed point theorem) was given by M. Hilden in 1977, [31].

A huge amount of literature has been produced towards extending Lomonosov's technique. One of the strongest results in this direction was proved by Lomonosov in 1991 [27]: let X be a Banach space, $\mathcal{A}$ be a proper subalgebra of $\mathcal{L}(X)$ (i.e. $\mathcal{A} \neq \mathcal{L}(X)$) which is weakly closed. Then there exist $x^{**} \in X^{**} \setminus \{0\}$ and $x^* \in X^* \setminus \{0\}$ such that $|x^{**}T^*x^*| \leqslant |||T|||$ for all $T \in \mathcal{A}$, where $|||T|||$ denotes the essential norm of T, i.e. its distance from the space of compact operators. A corollary of this result is the following: let X be a Banach space and $\mathcal{A}$ be a weakly closed proper subalgebra of $\mathcal{L}(X)$ such that there exist a net $\{A_\alpha\} \subseteq \mathcal{A}$ and a non-zero operator $A \in \mathcal{A}$ such that $A_\alpha^* \to A^*$ weakly and $|||A_\alpha||| \to 0$. Then $\{T^*\colon\ T \in \mathcal{A}\}$ is non-transitive. This corollary is a generalization of W. Burnside's theorem on matrix algebras: every proper algebra of matrices over an algebraically closed field has a non-trivial invariant subspace. The shortest proof of Burnside's theorem is given in [28].

In 1996, A. Simonič proved the following Hilbert space analogue of the 1991 Lomonosov's result [42]: let H be a complex Hilbert space, and $\mathcal{A}$ be a weakly closed proper subalgebra of $\mathcal{L}(H)$. Then there exist $x, y \in H \setminus \{0\}$ such that $|\mathrm{Re}\langle Tx, y\rangle| \leqslant |||\mathrm{Re}\, T|||\langle x, y\rangle$ for all $T \in \mathcal{A}$. As a corollary he obtained that every essentially selfadjoint operator (i.e. $T - T^*$ is a compact operator) on an infinite dimensional real Hilbert space has a non-trivial invariant subspace.

(C) Recall that a subnormal operator on a Hilbert space is the restriction of a normal operator to an invariant subspace. In 1978, S. Brown [14] proved that every subnormal operator has a non-trivial invariant subspace. Functional calculus was one of the main tools in the proof, which introduced the theory of dual algebras. For the developments on this theory see the book [12]. One of the main results produced by the theory of dual algebras is the 1986 result of Brown, Chevreau and Pearcy [16], that every contraction on the Hilbert space whose spectrum contains the unit circle has a non-trivial invariant subspace (also see [11]). Recall that an operator T on a Hilbert space is called hyponormal if $TT^* \leqslant T^*T$. Every subnormal operator is hyponormal. In 1987, Brown proved [15] that every hyponormal operator T has a non-trivial invariant subspace whenever $C(\sigma(T)) \neq R(\sigma(T))$ where for a compact $K \subset \mathbb{C}$, $C(K)$ denotes the continuous functions on K and $R(K)$ denotes the closure (in the $C(K)$ norm) of the rational functions on K with poles outside of K). The main tools of the proof are the theory of dual algebras and a result of [35] on properties of hyponormal operators.

(D) The technique of "extremal vectors" was introduced in 1998 by S. Ansari and P. Enflo [5]. Let X be a reflexive Banach space, T be an operator on X with dense range, $\varepsilon \in (0,1)$, $x_0 \in X$, $\|x_0\| = 1$. For every $n \in \mathbb{N}$ let $y_n \in T^{-n}\{x \in X\colon\ \|x_0 - x\| \leqslant \varepsilon\}$ with $\|y_n\| = \inf\{\|y\|\colon\ y \in T^{-n}\{x \in X\colon\ \|x_0 - x\| \leqslant \varepsilon\}\}$. Then (y_n) is called a sequence of extremal vectors of T with respect to ε and x_0. If x is a weak limit point of $(T^n y_n)_n$ and T satisfies additional assumptions it turns out that x is hyper-non-cyclic for T (i.e. x is non-cyclic for all operators commuting with T). Using the technique of minimal vectors, a special case of the 1973 result of Lomonosov is proved in [5]: let K be a non-zero compact operator on a Hilbert space. Then K has a non-trivial hyper-invariant subspace.

(E) In 2001, A. Atzmon and G. Godefroy defined that if X is a Banach space and $T \in \mathcal{L}(X)$, then T admits a moment sequence if there exist $x_0 \in X \setminus \{0\}$, $x_0^* \in X^* \setminus \{0\}$ and a positive Borel measure μ on $\mathbb{R}$ such that $x_0^*(T^n x_0) = \int u^n \, d\mu(u)$ for all $n \geqslant 0$, [8]. Then, using the smooth variational principle, they proved that every operator that admits a moment sequence on a real Banach space has a non-trivial invariant subspace.

(F) An example of a Banach space that admits an operator without any non-trivial invariant subspace (i.e. a transitive operator) was given by Enflo in the 70's, [18,19]. The technique was simplified in [9]. Further examples were given by C.J. Read, [37–40].

2. Fixed points of set-valued functions

Let $\mathcal{X}$ and $\mathcal{Y}$ be (Hausdorff) topological spaces, $\mathcal{P}(\mathcal{Y})$ denote the power set of $\mathcal{Y}$ and $\phi : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ be a set-valued function. The set-valued function ϕ is called lower semicontinuous (l.s.c.) if for any open subset V of $\mathcal{Y}$, the set $\{x \in \mathcal{X}\colon \phi(x) \cap V \neq \emptyset\}$ is open in $\mathcal{X}$. In terms of convergence of nets, this definition is equivalent to: for all $x \in \mathcal{X}$, for all $y \in \phi(x)$ and for all nets $(x_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{X}$ with $x_\lambda \to x$, we obtain that there exists $y_\lambda \in \phi(x_\lambda)$ such that $y_\lambda \to y$. The set-valued function ϕ is called upper semicontinuous (u.s.c.) if for any open set V of $\mathcal{Y}$, the set $\{x \in \mathcal{X}\colon \phi(x) \subseteq V\}$ is open in $\mathcal{X}$. In terms of convergence of nets, this definition is equivalent to: for all $x \in \mathcal{X}$, for all nets $(x_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{X}$ with $x_\lambda \to x$ and for all $y_\lambda \in \phi(x_\lambda)$ such that $(y_\lambda)_\lambda$ converges to some $y \in \mathcal{Y}$, we have that $y \in \phi(x)$. If $\mathcal{X} = \mathcal{Y}$ (i.e. $\phi : \mathcal{X} \to \mathcal{P}(\mathcal{X})$), then a point $x \in \mathcal{X}$ is a called a fixed point for ϕ if $x \in \phi(x)$. The following is the classical fixed point theorem for u.s.c. maps [17, Theorem 11.4].

Theorem 2.1. *Let C be a compact convex subset of a locally convex space $\mathcal{X}$ and let $\phi : C \to \mathcal{P}(C)$ be an u.s.c. set-valued map such that $\phi(x)$ is a non-empty closed convex set for all $x \in C$. Then ϕ has a fixed point.*

The corresponding fixed point theorem for l.s.c. maps requires that $\mathcal{X}$ is a Banach space [17, Theorem 11.6]:

Theorem 2.2. *Let C be a compact convex subset of a Banach space $\mathcal{X}$ and let $\phi : C \to \mathcal{P}(C)$ be an l.s.c. set-valued map such that $\phi(x)$ is a closed convex non-empty set for every $x \in C$. Then ϕ has a fixed point.*

For our purposes, we would not like to assume the metrizability of the space $\mathcal{X}$ in Theorem 2.2. We now explain how this can be done. Recall that if $\phi : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is a set-valued map, then a map $f : \mathcal{X} \to \mathcal{Y}$ is called a selection of ϕ if $f(x) \in \phi(x)$ for all $x \in \mathcal{X}$. For the set-valued map ϕ denote by $\overline{\phi}$ and $\overline{\mathrm{conv}}\,\phi$ (if the values of ϕ are subsets of a topological vector space) the maps defined by

$$\overline{\phi}(x) = \text{ the closure of the set } \phi(x);$$

$$\overline{\mathrm{conv}}\,\phi(x) = \text{ the closure of the convex hull of } \phi(x).$$

Recall that a (Hausdorff) topological space $\mathcal{X}$ is called paracompact if every open covering of $\mathcal{X}$ has a locally finite refinement. The classical Michael's selection theorem, [29, Theorem 3.2″], states that if $\mathcal{X}$ is a paracompact topological space, $\mathcal{Y}$ is a Banach space and $\phi:\mathcal{X}\to\mathcal{P}(\mathcal{Y})$ is an l.s.c. set-valued map whose values are non-empty closed convex subsets of $\mathcal{Y}$, then ϕ has a continuous selection. The assumption of lower semicontinuity can be weakened to "weak lower semicontinuity" as it was observed by K. Przeslawski and L. Rybinski (see [34] for details). The assumption that $\mathcal{Y}$ is a Banach space can also be weakened. In fact in [29] it was observed that the proof gives that $\mathcal{Y}$ can be assumed to be an F space (i.e. a complete metrizable topological vector space). The metrizability of $\mathcal{Y}$ is not necessary, since the result was further improved in [30] as follows:

Theorem 2.3. *Let $\mathcal{X}$ be a paracompact topological space, $\mathcal{Y}$ be a locally convex topological vector space with the property that the closed convex hull of any compact set is compact, and $\phi:\mathcal{X}\to\mathcal{P}(\mathcal{Y})$ be an l.s.c. map such that $\bigcup\{\phi(x)\colon x\in\mathcal{X}\}$ is metrizable and $\phi(x)$ is a complete set for all $x\in\mathcal{X}$. Then the map $\overline{\mathrm{conv}}\,\phi$ admits a continuous selection.*

For the proof of Theorem 2.2, one first uses the classical Michael's selection theorem in order to extract a continuous selection f of ϕ and then uses the Brouwer–Schauder–Tychonov's theorem (see for instance [4, Corollary 16.52]) in order to find a fixed point of f (which is a fixed point of ϕ as well). Following the same course of proof, but using Theorem 2.3 rather than the classical Michael's selection theorem we obtain

Theorem 2.4. *Let $\mathcal{X}$ be a locally convex TVS such that the closed convex hull of any compact set is compact. Let C be a compact convex metrizable subset of $\mathcal{X}$. Let $\phi:C\to\mathcal{P}(C)$ be an l.s.c. map. Then the map $\overline{\mathrm{conv}}\,\phi$ has a fixed point.*

Excellent references on set-valued maps, continuous selection and fixed point theorems are [24], [41] and [17], respectively.

3. Existence of invariant subspaces of operators

Now we show how to use set-valued maps in order to obtain the following result which is a special case of the result of [6].

Theorem 3.1. *Every compact operator on a complex reflexive Banach space has a non-trivial invariant subspace.*

Proof. Let X be a complex reflexive Banach space and T be a compact operator on X. It will be shown that T has a non-trivial invariant subspace. Without loss of generality assume that X is separable (else the closed linear span of the orbit of any non-zero vector under T is a non-trivial invariant subspace of T).

Let $x_0\in X$ with $\|x_0\|=1$. Set

$$A=\left\{x\in X\colon \|x_0-x\|\leqslant\frac{1}{2}\right\}\quad\text{and}\quad B=\left\{x^*\in X^*\colon x^*x_0=1\text{ and }\left\|x^*\right\|\leqslant 2\right\}.$$

Obviously A and B are non-empty convex, weakly closed and bounded sets. We separate two cases:

Case 1. There exists a sequence $(x_\ell)_{\ell\in\mathbb{N}} \in A^{\mathbb{N}}$ such that for every $m \in \mathbb{N}$ there exists $x_m^* \in B$ satisfying

$$\left|x_m^* T^k x_\ell\right| < \frac{1}{\ell} \quad \text{for all } 1 \leqslant k \leqslant \ell \leqslant m. \tag{1}$$

The reflexivity of X implies the weak compactness of the weakly closed convex bounded sets A and B. By the theorem of Eberlein–Šmulian, pass to subsequences of (x_ℓ) and (x_m^*) and relabel to assume that there exist $v \in A$ and $v^* \in B$ such that (x_ℓ) converges weakly to v and (x_m^*) converges weakly to v^*. Notice that (1) is still valid after the relabeling if one passes to the *same* subsequence of (x_ℓ) and $(x_m^*)_n$ before relabeling. Now fix $1 \leqslant k \leqslant \ell$ and let $m \to \infty$ in (1) to obtain

$$\left|v^* T^k x_\ell\right| \leqslant \frac{1}{\ell} \quad \text{for all } 1 \leqslant k \leqslant \ell. \tag{2}$$

Then fix $k \in \mathbb{N}$ and let $\ell \to \infty$ in (2) to obtain

$$v^* T^k v = 0 \quad \text{for all } k \in \mathbb{N}. \tag{3}$$

Since $0 \notin A$, we have $v \neq 0$. Thus, if $Tv = 0$ then $\overline{\operatorname{Span}} \operatorname{Orb}_T(v)$ is a non-trivial invariant subspace of T. If $Tv \neq 0$ then $\overline{\operatorname{Span}} \operatorname{Orb}_T(Tv)$ is a non-trivial invariant subspace of T (it is non-trivial since by (3) it is contained in the kernel of v^*, $\ker v^*$, and $\ker v^* \neq X$ since $v^* \neq 0$ since $0 \notin B$).

Case 2. For all $(x_\ell) \in A^{\mathbb{N}}$ there exists $m \in \mathbb{N}$ such that there is no $x^* \in B$ which satisfies $|x^* T^k x_\ell| < \frac{1}{\ell}$ for all $1 \leqslant k \leqslant \ell \leqslant m$.

Define a function $M : A^{\mathbb{N}} \to \mathbb{N}$ as follows: if $(x_\ell)_\ell \in A^{\mathbb{N}}$ then let $M((x_\ell)_\ell)$ be the smallest integer m such that there is no $x^* \in B$ which satisfies $|x^* T^k x_\ell| < \frac{1}{\ell}$ for all $1 \leqslant k \leqslant \ell \leqslant m$.

For $(x_\ell) \in A^{\mathbb{N}}$ define

$$Y\big((x_\ell)_\ell\big) = \operatorname{Span}\big\{T^k x_\ell\colon\ 1 \leqslant k \leqslant \ell \leqslant M\big((x_\ell)_\ell\big)\big\}. \tag{4}$$

Thus $Y((x_\ell)_\ell)$ is a finite dimensional subspace of X.

Claim 1. *For any sequence $(x_\ell)_\ell \in A^{\mathbb{N}}$ we have that $Y((x_\ell)_\ell) \cap A^o \neq \emptyset$, where A^o denotes the norm interior of A.*

For the proof of Claim 1 we will use the following elementary fact which appears as exercise in many introductory functional analysis books (and it is left as an exercise to the reader):

Fact. Let X be a Banach space, $x_0 \in X$, $\|x_0\| = 1$, and F be a finite dimensional subspace of F. Define the functional $x^* \in X^*$ by first defining it on $\operatorname{Span}\{x_0 \cup F\}$ by $x^*|_F = 0$ and $x^* x_0 = \operatorname{dist}(x_0, F)$ and then extend it to the whole space X by preserving its norm and keeping its name. Then $\|x^*\| = 1$.

Let $(x_\ell)_\ell \in A^{\mathbb{N}}$. By the assumption of Case 2 we know that there is no $x^* \in B$ which satisfies $|x^* T^k x_\ell| < \frac{1}{\ell}$ for all $1 \leqslant k \leqslant \ell \leqslant M((x_{\ell'})_{\ell'})$. By the definition of B we have that if $x^* \in X^*$ with $x^* x_0 = 1$ and $|x^* T^k x_\ell| < \frac{1}{\ell}$ for all $1 \leqslant k \leqslant \ell \leqslant M((x_{\ell'})_{\ell'})$ then $\|x^*\| > 2$. In particular, if $x^* \in X^*$ with $x^* x_0 = 1$ and $x^* T^k x_\ell = 0$ for all $1 \leqslant k \leqslant \ell \leqslant M((x_{\ell'})_{\ell'})$ then $\|x^*\| > 2$. By the above Fact, this implies that $\mathrm{dist}(x_0, Y((x_\ell)_\ell)) < \frac{1}{2}$, i.e. $Y((x_\ell)_\ell) \cap A^o \neq \emptyset$ which finishes the proof of Claim 1.

Note that the full strength of the assumption of Case 2 was not used. As a short parenthetical remark we explain the meaning of the assumption of Case 2. Let $(x_\ell)_\ell \in A^{\mathbb{N}}$ and let scalars $\{a_{k,\ell}\colon 1 \leqslant k \leqslant \ell \leqslant M((x'_\ell)_{\ell'})\}$ with $|a_{k,\ell}| < \frac{1}{\ell}$ for $1 \leqslant k \leqslant \ell \leqslant M((x'_\ell)_{\ell'})$. By the assumption of Case 2 we know that for every $x^* \in X^*$ with $x^*(x_0) = 1$ and $x^*T^k x_\ell = a_{k,\ell}$ for $1 \leqslant k \leqslant \ell \leqslant M((x'_\ell)_{\ell'})$ we have that $\|x^*\| > 2$. Thus for every $x^* \in X^*$ with $x^*(x_0) = 1$ and $x^*(T^k x_\ell - a_{k,\ell} x_0) = 0$ for $1 \leqslant k \leqslant \ell \leqslant M((x'_\ell)_{\ell'})$ we have that $\|x^*\| > 2$. Thus for every scalar $\{a_{k,\ell}\colon 1 \leqslant k \leqslant \ell \leqslant M((x'_\ell)_{\ell'})\}$ with $|a_{k,\ell}| < \frac{1}{\ell}$ for $1 \leqslant k \leqslant \ell \leqslant M((x'_\ell)_{\ell'})$ we have that

$$\mathrm{dist}\big(x_0, \mathrm{Span}\big\{T^k x_\ell - a_{k,\ell} x_0\colon 1 \leqslant k \leqslant \ell \leqslant M\big((x'_\ell)_{\ell'}\big)\big\}\big) < \frac{1}{2},$$

or equivalently,

$$\mathrm{Span}\big\{T^k x_\ell - a_{k,\ell} x_0\colon 1 \leqslant k \leqslant \ell \leqslant M\big((x'_\ell)_{\ell'}\big)\big\} \cap A^o \neq \emptyset.$$

This finishes the parenthetical remark of the explanation of the assumptions of Case 2. Now we return to our proof.

Define a set-valued function $\Phi : A^{\mathbb{N}} \to \mathcal{P}(X)$ by

$$\Phi\big((x_\ell)_\ell\big) = Y\big((x_\ell)_\ell\big) \cap A^o. \tag{5}$$

Notice that Claim 1 states that under the assumptions of Case 2, $\Phi((x_\ell)_\ell)$ is a non-empty set for all $(x_\ell)_\ell \in A^{\mathbb{N}}$.

For the remaining of the proof, endow X (and thus A) with the weak topology and $X^{\mathbb{N}}$ (and thus $A^{\mathbb{N}}$) with the product topology of the weak topology. Then, since X is reflexive, A and $A^{\mathbb{N}}$ are compact topological spaces.

Claim 2. *The function Φ defined in* (5) *is l.s.c.*

Assume for the moment that Claim 2 has been shown. Define

$$G : A^{\mathbb{N}} \to \mathcal{P}\big(A^{\mathbb{N}}\big) \quad \text{by } G\big((x_\ell)_\ell\big) = \big\{(x, x, \ldots)\colon x \in \Phi\big((x_\ell)_\ell\big)\big\}. \tag{6}$$

Obviously G is l.s.c. since Φ is l.s.c. Thus by Theorem 2.4, applied to $\mathcal{X} = X^{\mathbb{N}}$, $C = A^{\mathbb{N}}$ and $\phi = G$, we obtain that $\overline{G}$ has a fixed point. If $(x_\ell)_\ell \in A^{\mathbb{N}}$ is a fixed point of $\overline{G}$ then $(x_\ell)_\ell \in \overline{G}((x_\ell)_\ell)$. Thus there exists $a_0 \in A$ such that $x_\ell = a_0$ for all $\ell \in \mathbb{N}$. If $\mu = M((x_\ell)_\ell)$ then by the definition of $Y((x_\ell)_\ell)$, $\Phi((x_\ell)_\ell)$ is a finite linear combination of the vectors $\{T^k a_0\colon k = 1, \ldots, \mu\}$. Thus there exists a non-constant polynomial q such that $q(T)a_0 = a_0$, i.e. $1 \in \sigma_p(q(T))$. Since X is a complex Banach space, by the Fundamental Theorem of Algebra, $\sigma_p(q(T)) = q(\sigma_p(T))$ hence $\sigma_p(T) \neq \emptyset$, thus T has a non-zero eigenvector which spans a non-trivial invariant subspace of T. It only remains to establish Claim 2.

The proof of Claim 2 is based on the fact that the function $M : A^{\mathbb{N}} \to \mathbb{N}$ is an l.s.c. function. In order to see that M is an l.s.c. function, let $((x_{j,\ell})_{\ell\in\mathbb{N}})_{j\in\mathbb{N}} \subseteq A^{\mathbb{N}}$ and $(x_\ell)_{\ell\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $(x_{j,\ell})_{\ell\in\mathbb{N}} \to (x_\ell)_{\ell\in\mathbb{N}}$ as $j \to \infty$, i.e.

$$x_{j,\ell} \to x_\ell \text{ weakly as } j \to \infty, \text{ for all } \ell \in \mathbb{N}. \tag{7}$$

Let $\mu := M((x_\ell)_\ell)$ and for $j \in \mathbb{N}$, let $\mu_j := M((x_{j,\ell})_\ell)$. It is claimed that $\mu_j \geqslant \mu$ for all j large enough (which will finish the proof that the function M is an l.s.c. function). Indeed, if

$\mu_j \leqslant \mu - 1$ for infinitely many j's, then by passing to a subsequence and relabeling assume that $\mu_j \leqslant \mu - 1$ for all $j \in \mathbb{N}$.

Recall that by the definition of $M((x_\ell)_\ell)$, μ is the smallest integer m such that there is no $x^* \in B$ satisfying $|x^* T^k x_\ell| < \frac{1}{\ell}$ for all $1 \leqslant k \leqslant \ell \leqslant m$. Since $\mu - 1 < \mu$ we then have that there exists $b \in B$ such that

$$\left|bT^k x_\ell\right| < \frac{1}{\ell} \quad \text{for all } 1 \leqslant k \leqslant \ell \leqslant \mu - 1.$$

Then by (7) there exists j large enough such that

$$\left|bT^k x_{j,\ell}\right| < \frac{1}{\ell} \quad \text{for all } 1 \leqslant k \leqslant \ell \leqslant \mu - 1. \tag{8}$$

By the definition of $M((x_{j,\ell})_\ell)$,

$$\text{there is no } x^* \in B \text{ such that } \left|x^* T^k x_{j,\ell}\right| < \frac{1}{\ell} \text{ for all } 1 \leqslant k \leqslant \ell \leqslant \mu_j. \tag{9}$$

Notice that (8), (9) and the fact that $\mu_j \leqslant \mu - 1$ give a contradiction. This contradiction proves that $\mu_j \geqslant \mu$ for all j large enough, and by passing to a subsequence and relabeling, assume that $\mu_j \geqslant \mu$ for all $j \in \mathbb{N}$.

The proof of Claim 2 continues as follows: let $((x_{j,\ell})_{\ell\in\mathbb{N}})_{j\in\mathbb{N}} \subseteq A^{\mathbb{N}}$ and $(x_\ell)_{\ell\in\mathbb{N}} \in A^{\mathbb{N}}$ satisfying (7), $\mu := M((x_\ell)_\ell)$, $\mu_j := M((x_{j,\ell})_\ell)$ (for $j \in \mathbb{N}$), and $y \in \Phi((x_\ell)_\ell)$. By (4) and (5) there exist $(a_{k,\ell})_{\ell=1,\,k=1}^{m,\ell} \subset \mathbb{C}$ such that

$$y = \sum_{\ell=1}^{\mu} \sum_{k=1}^{\ell} a_{k,\ell} T^k x_\ell.$$

For $j \in \mathbb{N}$ let $Y_j := Y((x_{j,\ell})_\ell)$ and

$$y_j := \sum_{\ell=1}^{\mu} \sum_{k=1}^{\ell} a_{k,\ell} T^k x_{j,\ell}.$$

Then, for all j's large enough, $\mu_j \geqslant \mu$ (since the function M is an l.s.c. function), and thus $y_j \in Y_j$. By (7) and the compactness of T, we obtain that $y_j \to y$ in norm. Since $y \in A^o$, we obtain that for j large enough we also have that $y_j \in A^o$ and hence $y_j \in \Phi((x_{j,n})_{n\in\mathbb{N}})$. Since $y_j \to y$ weakly as $j \to \infty$, the set-valued function Φ is l.s.c. which finishes the proof of Claim 2. □

Notice that in the proof of Theorem 3.1, the compactness of the operator T was only used in the fourth line before the end of proof. Thus the compactness of T was used to show that the map Φ defined in (5) is l.s.c. In fact, in the fourth line before the end of the proof of Theorem 3.1, it is obtained that "$y_j \to y$ *in norm*" which implies that the function Φ is l.s.c. *even if the range space is equipped with the norm topology* (while still the domain is equipped with the product of the weak topology). Note that for the proof of Theorem 3.1 to work, it is only required that Φ is l.s.c. when the range space is equipped with the weak topology. Thus it is reasonable to ask whether the assumption of the compactness of the operator in the above proof can be omitted, or replaced by a weaker assumption. In the next paragraph we make this question more precise.

In the proof of Theorem 3.1 notice that for the compact operator T acting on a reflexive Banach space X we proved in Case 2 that the function G (defined in (6)) is l.s.c. Then Theorem 2.4

implied that $\overline{G}$ has a fixed point. The fixed point of $\overline{G}$ immediately implied the existence of an eigenvalue of T. Now consider classes of operators T acting on a reflexive Banach space X other than the compact operators. For such T if we can prove in Case 2 that $\overline{G}$ has a fixed point then the above proof will show that T has a non-trivial invariant subspace. Notice that in order to show that $\overline{G}$ has a fixed point, it is enough to construct a set-valued function $\psi : A^{\mathbb{N}} \to \mathcal{P}(A^{\mathbb{N}})$ such that $\psi \subseteq \overline{G}$ (i.e. $\psi((x_\ell)_\ell)$ is a subset of the closure of $G((x_\ell)_\ell)$ for all $(x_\ell)_\ell \in A^{\mathbb{N}}$), and show that ψ has a fixed point. Indeed, any fixed point of such ψ will be a fixed point of $\overline{G}$ as well. The main tools for extracting fixed points of set-valued functions are Theorems 2.4 and 2.1. In order to be able to apply to a function $\psi \subseteq \overline{G}$, Theorem 2.4, ψ has to be l.s.c. and for Theorem 2.1, ψ has to be u.s.c. with convex closed values. These observations motivate us to ask the following two questions which suggest possible further uses of the method presented in this article. By our above discussion, affirmative answer to any of these questions implies existence of a non-trivial invariant subspace of the operator.

Question 3.2. For which classes of operators T acting on a reflexive Banach space X, under the assumptions of Case 2, there exists an l.s.c. set-valued function $\psi : A^{\mathbb{N}} \to \mathcal{P}(A^{\mathbb{N}})$ such that $\psi \subseteq \overline{G}$?

Question 3.3. For which classes of operators T acting on a reflexive Banach space X, under the assumptions of Case 2, there exists an u.s.c. set-valued function $\psi : A^{\mathbb{N}} \to \mathcal{P}(A^{\mathbb{N}})$ such that $\psi \subseteq \overline{G}$ and $\psi((x_\ell)_\ell)$ is a convex closed set for all $(x_\ell)_\ell \in A^{\mathbb{N}}$?

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