24 June 2007
Math 242

Exercises on Laplace Transforms

Directions: Please complete the following problems for homework. This will be due on Thursday, June 28. Please note that problem 4 is a bonus problem and not required.

1. Use the *definition* of the Laplace transform to calculate the following Laplace transforms:

(a) $f(t) = t^2$

(b) $f(t) = te^{3t}$

(c) $f(t) = \begin{cases} 15 & 0 \le t \le 4 \\ 3t + 2 & t \ge 4 \end{cases}$

2. Show that

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}$$

for any real number k and appropriate s. Remember that we computed $\mathcal{L}\{\sin(kt)\}$ in class; this computation is similar.

3. Calculate the following Laplace transforms using the formulas we learned in class. DO NOT use the definition to compute these!

(a) $f(t) = (1 + e^{2t})^2$

(b) $f(t) = 4t^2 - 5\sin 3t$

4. Use the translation theorems to compute the Laplace transform of the following functions: (a) $f(t) = 2te^{4t}$ (b) $f(t) = t^6 e^{-19t}$ (c) $f(t) = e^{5t}\cos(2t)$ (d) $f(t) = (t - 1)H(t - 1)$ (e) $f(t) = \cos(3t)H(t - \pi)$

4 (Bonus Problem). Define the *gamma function* as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

when $x > 0$.

(a) Show that for any real number $\alpha > -1$ we have

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Note that this is a generalization of the formula we discussed in class. The formula from class works only for integers n and this works for any real number larger than -1. When $\alpha = n$ is an integer, one can show that $\Gamma(n+1) = n!$. Thus, the whole point of the gamma function is that it generalizes the factorial function.

(b) It can be shown that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Use this fact and the formula $\Gamma(x+1) = x\Gamma(x)$ to compute the Laplace transform of the following: (a) $t^{-1/2}$ (b) $\sqrt{t}$ (c) $t^{3/2}$