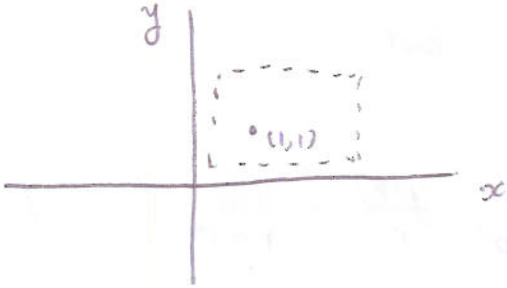
Solutions to Homework 2

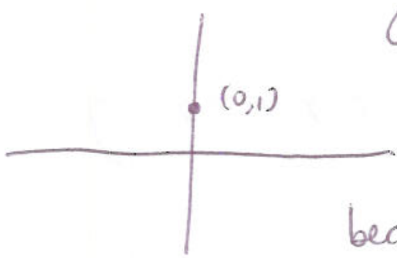
1. (a) We have $\begin{cases} \frac{dy}{dx} = \frac{y}{x} \\ y(x_0) = y_0 \end{cases}$ so $f(x,y) = \frac{y}{x}$ and $\frac{\partial f}{\partial y} = \frac{1}{x}$

These are continuous at each (x,y) so that $x \neq 0$. If $x_0 = 1$ and $y_0 = 1$, we may draw a rectangle containing $(1,1)$ so that f and $\frac{\partial f}{\partial y}$ are continuous therein

We may draw any rectangle containing $(1,1)$



The theorem applies and there is a local solution. If $x_0 = 0$ and $y_0 = 1$ the theorem does not apply. Why? Draw a picture



One cannot draw a rectangle containing $(0,1)$ so that f and $\frac{\partial f}{\partial y}$ are continuous because both f and $\frac{\partial f}{\partial y}$ blow up when $x = 0$. The theorem therefore does not apply here.

(b) We want to solve $\begin{cases} xy' = y \\ y(3) = 5 \end{cases}$ We have $\int \frac{dy}{y} = \int \frac{dx}{x}$

so $\ln y = \ln x + c$ (don't worry about absolute values).

Then $y = c_1 x$ (here, $c_1 = e^c$). $y(3) = 5$ so

$y(3) = 3c_1 = 5$ so $c_1 = \frac{5}{3}$ and $y = \frac{5}{3}x$

(c) Since $0 \geq 0$, we have $y(0) = 0$. For it to satisfy $\begin{cases} xy' = y \\ y(0) = 0 \end{cases}$, it must be differentiable at 0 (we are ok away from 0). We have

$$y'(0) = \lim_{x \to 0} \frac{y(x) - y(0)}{x - 0} = \lim_{x \to 0} \frac{y(x)}{x}. \quad \text{But}$$

$$\lim_{x \to 0^-} \frac{y(x)}{x} = \lim_{x \to 0^-} \frac{0}{x} = 0 \quad \text{and} \quad \lim_{x \to 0^+} \frac{y(x)}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

so $y'(0)$ does not exist. In $(1,4)$, $y(x) = x$

Clearly $y(3) = 3$ and $y' = 1$ so $xy' = x \cdot 1 = x = y$

so we do satisfy $\begin{cases} xy' = y \\ y(3) = 3 \end{cases}$ on $(1,4)$

2. Our model is $\begin{cases} \frac{dP}{dt} = kP \\ P(0) = 50{,}000 \end{cases}$ and $P(1) = 51{,}794$.

We have $\frac{dP}{P} = k\,dt$ so $\ln P = kt + C_0$ and

$P(t) = Ce^{kt}$. $P(0) = C = 50{,}000$ so $P(t) = 50{,}000e^{kt}$.

Now, $P(1) = 50{,}000\, e^{k} = 51{,}794$ so

$e^{k} = 1.03588$ so $k = 0.0352513\ldots$

and $P(t) = 50{,}000\, e^{0.0352513\, t}$. $P(10) \approx 71131.91$

or about 71,132 people.

3 (a). Separate to get $\int \frac{dy}{y^2+1} = \int 4\, dx$ so

$\tan^{-1} y = 4x + c$ and $y = \tan(4x+c)$. Now,

$y(\frac{\pi}{4}) = \tan(\pi + c) = 1$ so $\pi + c = \frac{\pi}{4}$ so $c = -\frac{3\pi}{4}$

and $\boxed{y = \tan\left(4x - \frac{3\pi}{4}\right)}$

(b) The problem should read $\begin{cases} x^2 y' = y - xy \\ y(-1) = -1. \end{cases}$

We have $x^2 y' = y(1-x)$ so $\int \frac{dy}{y} = \int \frac{1-x}{x^2}\, dx$

$$\ln y = -\frac{1}{x} - \ln x + c_0$$

so $y = c e^{-\frac{1}{x} - \ln x} = \frac{c}{e^{1/x} \cdot x}$. Then

$y(-1) = \frac{c}{e^{-1}(-1)} = -1$ so $c = e^{-1}$.

(c) We have $\frac{dx}{dt} = 1-2x$ so $\underbrace{\int \frac{dx}{1-2x}}_{\substack{u=1-2t \\ du=-2dt}} = \int dt$

so $-\frac{1}{2}\ln(1-2x) = t + c_0$ or $\ln(1-2x) = -2t + c_1$

and $1-2x = ce^{-2t}$ so $\frac{1-ce^{-2t}}{2} = x(t)$

$x(0) = \frac{1-c}{2} = \frac{5}{2}$ so $1-c=5$ and $c=-4$ so

$$x(t) = \frac{1+4e^{-2t}}{2} = \frac{1}{2} + 2e^{-2t}.$$