28 February 2008
Math 242

Solutions to Homework 4 and 5

Please note that I graded numbers 2, 20, and 35.

2. Solve $2xyy' = x^2 + 2y^2$.

We first isolate y' to get

$$y' = \frac{x}{2y} + \frac{y}{x} = \frac{1}{2(y/x)} + \frac{y}{x}$$

Letting $F(v) = \dfrac{1}{2v} + v$, we see that $y' = F\left(\dfrac{y}{x}\right)$. Set $v = (y/x)$ so that $y = vx$ and $y' = v'x + v$ and $v'x + v = \dfrac{1}{2v} + v$ or $v'x = \dfrac{1}{2v}$. Separating variables gives

$$\int 2v\ dv = \int \frac{dx}{x}$$

or $v^2 = \ln x + c$ or $y^2 = x^2(\ln x + c)$.

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20. Solve $y^2y' + 2xy^3 = 6x$.

As usual, we first isolate y' to get $y' + 2xy = 6xy^{-2}$ (*). This equation is Bernoulli where $n = -2$. Then $v = y^{1-n} = y^3$ so that $v' = 3y^2y'$. Multiplying both sides of (*) by $3y^2$ gives $3y^2y' + 2xy(3y^2) = 6xy^{-2}(3y^2)$ or, after simplification, $v' + 6xv = 18x$. Then

$$\mu(x) = \exp\left(\int 6x\ dx\right) = e^{3x^2}$$

so that $(ve^{3x^2})' = 18xe^{3x^2}$ so that

$$ve^{3x^2} = 18\int xe^{3x^2}dx = 3e^{3x^2} + C$$

Note that the integral above may be computed by letting $u = 3x^2$. We then see that $v = y^3 = 3 + Ce^{-3x^2}$.

$\Diamond$

35. Solve $\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x)dy = 0$

We first test to see that this is exact. Let $M(x, y) := x^3 + \dfrac{y}{x}$ and $N(x, y) := y^2 + \ln x$. Then

$$\frac{\partial M}{\partial y} = \frac{1}{x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{1}{x}$$

so the equation is exact. Now we solve; we want to find an $f(x, y)$ so that $f_x = M$ and $f_y = N$ (f_x and f_y are shorthand notations for partial derivatives with respect to x and y). Then $f_x = M = x^3 + \frac{y}{x}$ tells us that $f(x, y) = \frac{1}{4}x^4 + y \ln x + g(y)$ after a simple integration. But $f_y = \ln x + g'(y)$ so $f_y = N = y^2 + \ln x = \ln x + g'(y)$ implies that $g'(y) = y^2$ or $g(y) = \frac{1}{3}y^3$. The solution is therefore

$$f(x, y) = \frac{1}{4}x^4 + \frac{1}{3}y^3 + y \ln x = c$$

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