Name (print legibly): **Solutions**
19 April 2007
Math 142

Quiz 9

Directions: You have 20 minutes to complete the quiz. Please show all relevant steps - if you show no work, I cannot give you partial credit. No calculators are allowed during the quiz. This quiz is worth 15 points.

1. (3 Points) For what values of x does the series $\sum_{k=0}^{\infty}(-1)^k x^{2k}$ converge? Find it's sum for these values of x (*Hint:* Note that $(-1)^k x^{2k} = (-x^2)^k$).

Solution. Recall that whenever $|x| < 1$, we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

It easily follows that

$$\sum_{k=0}^{\infty}(-1)^k x^{2k} = \sum_{k=0}^{\infty}(-x^2)^k = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

This series converges when $|x^2| < 1$ or $|x| < 1$ (it's geometric with $a = 1$ and $r = -x^2$).

2. (12 Points) Test the series for convergence by using the given test.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ (Integral Test)

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2}{k^3+1}$ (Limit Comparison Test)

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{11^k}{3k!}$ (Ratio Test)

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ (Comparison Test)

Solution. For part (a), we note that letting $u = x^2 + 1$ gives

$$\begin{aligned} \int_1^{\infty} \frac{x}{x^2+1} dx &= \lim_{h\to\infty} \int_1^{h} \frac{x}{x^2+1} dx = \lim_{h\to\infty} \frac{1}{2} \ln(x^2+1)\Big|_1^h \\ &= \frac{1}{2} \lim_{h\to\infty} \left(\ln(h^2+1) - \ln 2\right) = +\infty \end{aligned}$$

Then the sum diverges by the integral test. For part (b), let $a_k = \frac{k^2}{k^3+1}$ and $b_k = \frac{1}{k}$ so that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1$$

By the limit comparison test, both $\sum a_k$ and $\sum b_k$ either both converge or diverge. But $\sum_{k=1}^{\infty} \frac{1}{k}$ is a harmonic series and diverges so that $\sum b_k$ diverges. For part (c), let $a_k = 11^k/(3k!)$ so that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{11^k \cdot 11}{3(k+1)k!} \cdot \frac{3k!}{11^k} = \lim_{k\to\infty} \frac{11}{k+1} = 0$$

By the ratio test, the series converges. Part (d) was an example from class. Note that $1 < \ln k$ whenever $k \geq 3$ so that when $k \geq 3$ we get

$$\frac{1}{k} < \frac{\ln k}{k}$$

The sum of the left hand side above diverges so the sum of the right hand side above diverges by the comparison test.