Seminar Notes: On the irreducibility of a truncated binomial expansion

(joint work with Dmitrii V. Pasechnik)

Notation:

- n and k are positive integers with $k \le n-1$
- p and q are primes (to be chosen $> k$)
- a_j are integers having no prime factors $> k$ (and, hence, non-zero)
- $P_{n,k}(x) = \sum_{j=0}^{k} \binom{n}{j} x^j.$
- $c_j = \dfrac{(-1)^{k-j} n(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k+1)(n-k)}{j!(k-j)!}$
- $F_{n,k}(x) = \sum_{j=0}^{k} a_j c_j x^j$

Main Results:

Theorem 1. *Let N be a positive integer. For each integral pair (n,k) with $1 \le n \le N$ and $1 \le k \le n-2$, consider the set $S(n,k)$ of all polynomials of the form $F_{n,k}(x)$. The number of such pairs (n,k) for which there exists a polynomial $f(x) \in S(n,k)$ that is reducible is $\ll N^{1.525}$.*

Theorem 2. *If there is a prime $p > k$ that exactly divides $n(n-k)$, then $F_{n,k}(x)$ is irreducible.*

Theorem 3. *For $k \ge 3$, there is an $n_0 = n_0(k)$ such that if $n \ge n_0$, then $F_{n,k}(x)$ is irreducible.*

Important Identities:

- $\sum_{j=0}^{a} \binom{b}{j} (-1)^j = \binom{b-1}{a} (-1)^a \quad$ for $0 \le a \le b$
- $P_{n,k}(x-1) = \sum_{i=0}^{k} \binom{n}{i} \binom{n-i-1}{k-i} (-1)^{k-i} x^i = \sum_{j=0}^{k} c_j x^j$

The Lemmas (for Theorem 3):

Lemma 1. *Let p be a prime $> k$ and e a positive integer for which $\nu_p(n) = e$ or $\nu_p(n-k) = e$. Then each irreducible factor of $f(x)$ has degree a multiple of $k/\gcd(k,e)$.*

Lemma 2. *Let n' be the largest divisor of $n(n-k)$ that is relatively prime to $k!$. Write*

$$n' = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

where the p_j denote distinct primes and the e_j are positive integers. Let

$$d = \gcd(k, e_1, e_2, \ldots, e_r).$$

Then the degree of each irreducible factor of $f(x)$ is a multiple of k/d.

Lemma 3. *Let $f(x) = F_{n,k}(x)$. Let n'' be the largest divisor of $(n-1)(n-k+1)$ relatively prime to $k!$. Suppose $\nu_p(n'') = e$ where $p > k$ and $e \in \mathbb{Z}^+$. If $f(x)$ is a product of two polynomials of degree $k/2$, then $(k-1)|e$.*