Irreducibility of Classical Polynomials and their Generalizations

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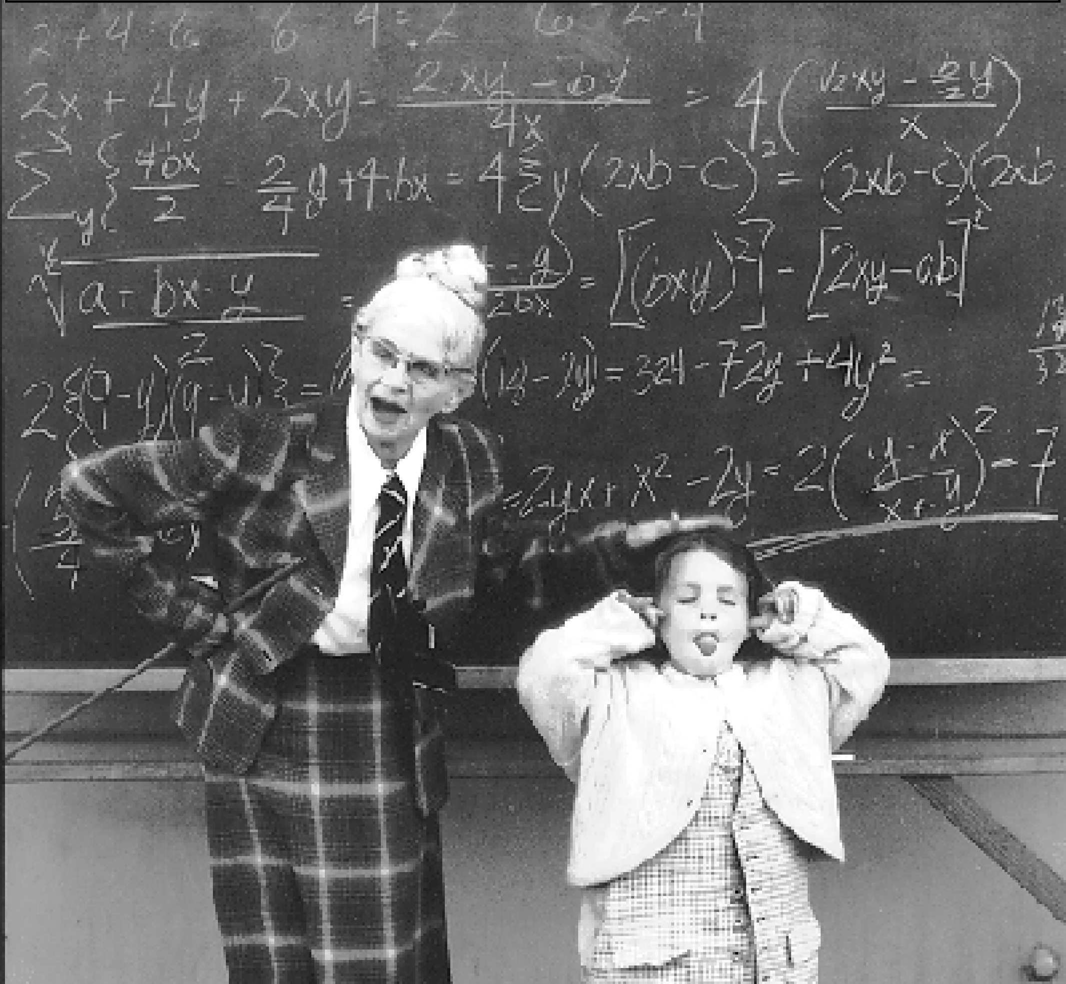
This talk is dedicated to John Brillhart and his influence on my mathematics.

En route to a conference, we stop at the San Diego Zoo.

At a conference in Assilomar.

John at a young age refusing to learn his mathematics.



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- $f(x)$ has degree at least 1,
- $f(x)$ does not factor as a product of two polynomials in $\mathbb{Q}[x]$ each of degree ≥ 1.

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Result (due to A. Cohn): If $f(x)$ has digits as coefficients and $f(10)$ is prime, then $f(x)$ is irreducible.

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Result (due to A. Cohn): If $f(x)$ has digits as coefficients and $f(\mathbf{10})$ is prime, then $f(x)$ is irreducible.

Theorem: The analogous result holds in any base $b \geq \mathbf{2}$.

Theorem: Let $f(x) = \sum_{j=0}^{n} a_j x^j$ with $0 \le a_j < 10$ and $f(10)$ prime. Then $f(x)$ is irreducible.

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Comment: There exist polynomials $f(x) \in \mathbb{Z}[x]$ with non-negative coefficients with $f(10)$ prime and with $f(x)$ reducible.

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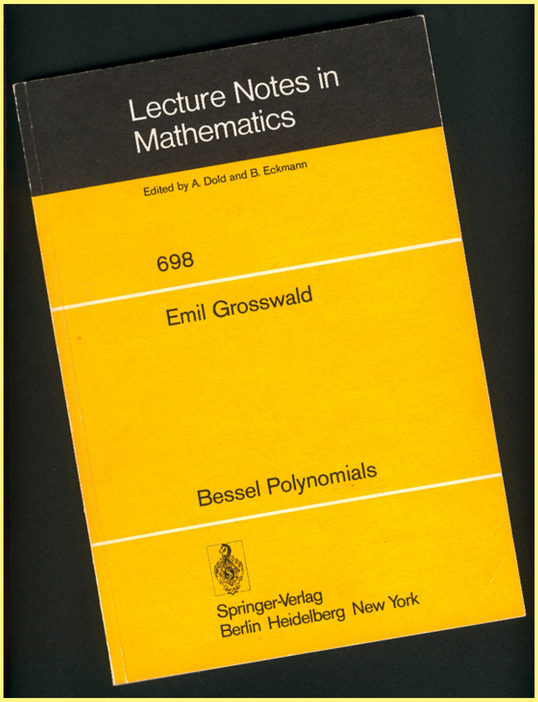

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Remark: In joint work with Ognian Trifonov this conjecture has now been resolved in the affirmative (to appear).

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  - Calvin and Hobbes

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$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n} = \sum_{j=0}^{n} \frac{(-1)^j}{j!}\binom{n}{j}x^j$$

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Theorem 1 (I. Schur, 1929): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n\frac{x^n}{n!} + a_{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots + a_1 x + a_0$$

is irreducible.

Theorem (1996): Let $a_0, a_1, \ldots, a_n$ denote arbitrary integers with $|a_0| = 1$, and let

$$f(x) = \sum_{j=0}^{n} a_j x^j / j!.$$

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in which cases either $f(x)$ is irreducible or $f(x)$ is the product of two irreducible polynomials of equal degree. If $|a_n| = n$, then for some choice of $a_1, \ldots, a_{n-1} \in \mathbb{Z}$ and $a_0 = \pm 1$, we have that $f(x)$ is divisible by $x \pm 1$.

The Generalized Laguerre Polynomials:

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$$
\begin{aligned}
L_n^{(\alpha)}(x) &= \frac{e^x x^{-\alpha}}{n!} \frac{d^n\left(x^{n+\alpha}e^{-x}\right)}{dx^n} \\
&= \sum_{j=0}^{n} \frac{(n+\alpha)\cdots(j+1+\alpha)(-x)^j}{(n-j)!j!}
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$L_n^{(0)}(x) = L_n(x)$ (the Laguerre Polynomials)

$$\boldsymbol{L_n^{(1)}(x) = (n+1)\sum_{j=0}^{n}\binom{n}{j}\frac{(-x)^j}{(j+1)!}}$$

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Theorem 2 (I. Schur): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n\frac{x^n}{(n+1)!} + a_{n-1}\frac{x^{n-1}}{n!} + \cdots + a_1\frac{x}{2} + a_0$$

is irreducible (over the rationals) unless $n = 2^r - 1 > 1$ (when $x \pm 2$ can be a factor) or $n = 8$ (when a quadratic factor is possible).

Theorem (joint with M. Allen): For n an integer ≥ 1, define

$$f(x) = \sum_{j=0}^{n} a_j \frac{x^j}{(j+1)!}$$

where the a_j's are arbitrary integers with $|a_0| = 1$. Write

$$n + 1 = k' 2^u \quad \text{with } k' \text{ odd}$$

and

$$(n+1)n = k'' 2^v 3^w \quad \text{with } \gcd(k'', 6) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then $f(x)$ is irreducible.

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$$\begin{aligned}
L_2^{(2)}(x) &= \frac{1}{2}(x-2)(x-6) \\
L_2^{(23)}(x) &= \frac{1}{2}(x-20)(x-30) \\
L_4^{(23)}(x) &= \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040) \\
L_4^{(12/5)}(x) &= \frac{1}{15000}(25x^2-420x+1224)(25x^2-220x+264) \\
L_5^{(39/5)}(x) &= \frac{-1}{375000}(5x-84)\big(625x^4-29500x^3 \\
&\qquad +448400x^2-2662080x+5233536\big)
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Theorem (joint with T.-Y. Lam): Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers n, the polynomial $L_n^{(\alpha)}(x)$ is irreducible over the rationals.

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A Special Case: $\alpha = n$

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- Hilbert's work and work of E. Noether (1918) began what has come to be known as Inverse Galois Theory.
- Van der Waerden showed that for "almost all" polynomials $f(x) \in \mathbb{Z}[x]$, the Galois group associated with $f(x)$ is the symmetric group S_n.

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- Schur showed $\displaystyle\sum_{j=0}^{n} \frac{x^j}{j!}$ has Galois group A_n if $4|n$.
- Schur did not find an explicit sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod 4$.

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Comment: Gow also showed that $L_n^{(n)}(x)$ is irreducible if

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Conjecture: If $n > 2$, then $L_n^{(n)}(x)$ is irreducible.

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible (and, hence, has Galois group A_n for almost all $n \equiv 2 \pmod 4$). More precisely, the number of $n \le t$ such that $L_n^{(n)}(x)$ is reducible is

$$\ll \exp\left(\frac{9\log(2t)}{\log\log(2t)}\right).$$

Furthermore, for all but finitely many n, $L_n^{(n)}(x)$ is either irreducible or $L_n^{(n)}(x)$ is the product of a linear polynomial times an irreducible polynomial of degree $n-1$.

Theorem (joint work with R. Williams): For all but

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positive integers $n \le t$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j \binom{2n}{n-j} \frac{x^j}{j!}$$

is irreducible over the rationals for every choice of integers $a_0, a_1, \dots, a_n$ with $|a_0| = |a_n| = 1$.

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Comment: The number of $n \le t$ for which $f(x)$ is reducible for some choice of a_j as above is

$$\gg \log t.$$

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However, in joint work with O. Trifonov (and input from R. Tijdeman, F. Beukers, and M. Bennett), the argument can now be made effective. What's needed is:

> *There exist explicit numbers α and $\beta > 0$ such that, for $n \geq \alpha$,*
>
> $$n(n+1) = 2^k 3^\ell m \implies m > n^\beta.$$

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The Ramanujan-Nagell equation

$$x^2 + 7 = 2^n$$

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Moreover, there exist explicit numbers α and $\beta > 0$ such that, for $x \geq \alpha$,

$$x^2 + 7 = 2^n m \implies m \geq x^\beta.$$

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$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n \left(e^{-x^2/2}\right)}{dx^n}$$
$$= \sum_{j=0}^{[n/2]} (-1)^n \binom{n}{2j} u_{2j}\, x^{m-2j}$$

where

$$u_{2j} = (2j-1)(2j-3)\cdots 3 \cdot 1$$

The Hermite Polynomials:

$$H_{2n}(x) = (-1)^n u_{2n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{2j}}{u_{2j}}$$

$$H_{2n+1}(x) = (-1)^n u_{2n+2} x \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{2j}}{u_{2j+2}}$$

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Theorem 3 (I. Schur, 1929): For $n > 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

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Theorem 4 (I. Schur, 1929): For $n \geq 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j x^{2j} / u_{2j+2}$$

is irreducible unless $2n$ is of the form $3^u - 1$ with $u > 1$.

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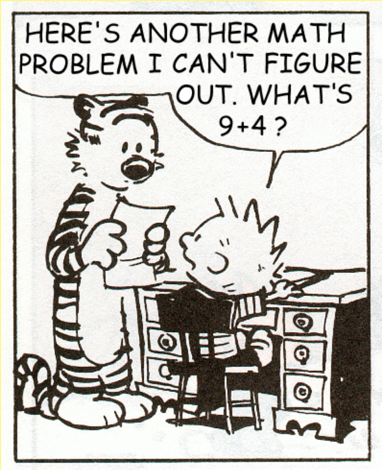

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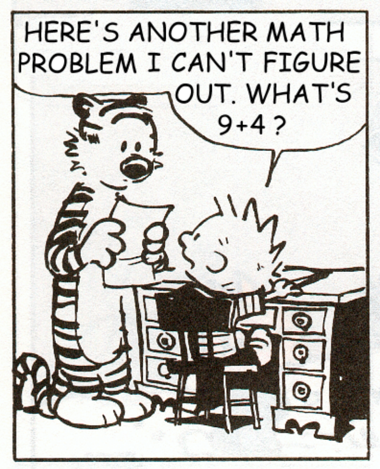

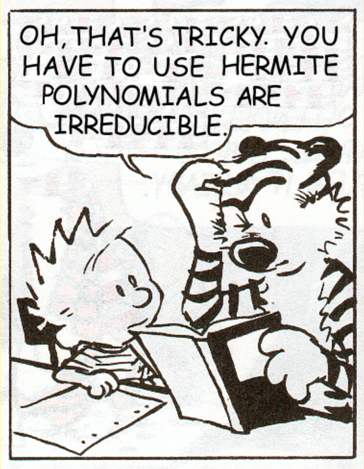

Email from Mark Kon:

Given a function $f(x)$, its wavelet transform consists of the family of functions $g(2^j x) * f(x)$, where g is the gaussian function, and j is an integer. The question was: if we know the zeroes of the second derivatives of this family of functions (over all j), can we recover f? ... The problem reduces to showing that none of these polynomials [certain Hermite polynomials] has zeroes (aside from the trivial one at the origin) which coincides with a zero of another one. So the bottom line is that the conjecture that f is uniquely recoverable follows from the non-overlapping of the zeroes of the Hermite polynomials.

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A Similar Type Application of Irreducibility:

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Then $p(x)$ is $(x-1)^3$ times an irreducible polynomial if n is even and $(x-1)^3(x+1)$ times an irreducible polynomial if n is odd.

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Joint Work With A. Borisov, T.-Y. Lam, O. Trifonov:

True for all but $O(t^{4/5+\varepsilon})$ *values of* $n \leq t$.

Theorem 3 (I. Schur, 1929): For $n > 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j x^{2j} / u_{2j}$$

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Theorem (joint with M. Allen): For $n > 1$ and arbitrary integers a_j with $|a_0| = 1$ and

$$0 < |a_n| < 2n - 1,$$

the polynomial $f(x)$ above is irreducible for all but finitely many pairs (a_n, n).

Theorem 4 (I. Schur, 1929): For $n \geq 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j x^{2j} / u_{2j+2}$$

is irreducible unless $2n$ is of the form $3^u - 1$ with $u > 1$.

Theorem (joint with M. Allen): For n an integer ≥ 1, define

$$f(x) = \sum_{j=0}^{n} a_j \frac{x^{2j}}{u_{2j+2}}$$

where the a_j's are arbitrary integers with $|a_0| = 1$. Write

$$2n+1 = k' 3^u \quad \text{with} \quad 3 \nmid k'$$

and

$$(2n+1)(2n-1) = k'' 3^v 5^w \quad \text{with} \quad (k'', 15) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then $f(x)$ is irreducible for all but finitely many pairs (a_n, n).

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Theorem (joint with O. Trifonov): If $a_0, a_1, \dots, a_n$ are arbitrary integers with $|a_0| = |a_n| = 1$, then

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A result of M.G. Dumas (in 1906) eliminates possible degrees for the factors of a polynomial using information about the divisibility of the coefficients by a given prime p (forming Newton polygons with respect to p).

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

> "Two such factorization schemes with a common, non-trivial factorization, will be called *compatible*. Otherwise, we call them incompatible. It is clear that if one can exhibit two incompatible factorization schemes, one thereby will have proved the irreducibility of the polynomial considered."
>
> Emil Grosswald
> Bessel Polynomials
> Lecture Notes Series

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

  Idea: To consider factorization schemes using many primes and show that they are incompatible. For a polynomial of degree n and a $k \in [1, n/2]$, find a prime p such that the Newton polygon with respect to p does not allow for a factor of $f(x)$ to have degree k.

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$$\prod_{\substack{p^r || n(n-1)\cdots(n-k+1) \\ p \ge k+1}} p^r > n.$$

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For n large and k large (say $> n^{2/3}$), use that there are two primes in the interval $[n-k+1, n]$.

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Now consider k small.

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Lemma. For $3 \le k \le n/2$,

$$\prod_{\substack{p^r || n(n-1)\cdots(n-k+1) \\ p \ge k+1}} p^r > n$$

unless one of the following holds:

$$\begin{array}{lll} k = 3 & \text{and} & n = 6, 8, 9, 10, \text{ or } 18 \\ k = 4 & \text{and} & n = 9 \\ k = 5 & \text{and} & n = 10 \text{ or } 12. \end{array}$$

The Generalized Bernoulli Polynomials:

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J. Brillhart's Observation (1969):

$$\begin{aligned} 6B_{11}(x) = {} & x(x-1)(2x-1)(x^2-x-1) \\ & \times (3x^6-9x^5+2x^4+11x^3+3x^2-10x-5) \end{aligned}$$

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Theorem (joint with A. Adelberg): A positive proportion of the polynomials $B_n^{(n)}(x)$ are Eisenstein (and, hence, irreducible).

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Theorem (joint with A. Adelberg): A positive proportion of the polynomials $B_n^{(n)}(x)$ are Eisenstein (and, hence, irreducible). More precisely, if the number of $n \le t$ for which $B_n^{(n)}(x)$ is Eisenstein is $\mathcal{B}(t)$, then

$$\mathcal{B}(t) > t/5 \qquad \text{for } t \text{ sufficiently large.}$$

Time for Questions