On the Factorization of

$f(x)x^n + g(x)$

Michael Filaseta and Manton Matthews, Jr.

University of South Carolina

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ever end?

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$\boldsymbol{x}^{\mathbf{3}}$ is the least power of $\boldsymbol{x}$ that can be added to $\mathbf{1}$ to get a reducible polynomial

$\boldsymbol{x}^{\mathbf{15}}$ is the *next* power of $\boldsymbol{x}$ that can be added to $\mathbf{1} + \boldsymbol{x}^{\mathbf{3}}$ to get a reducible polynomial

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Answer: [ ]

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$$\uparrow$$

$$1$$

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For general $f(x)$ and $g(x)$, there may be some.

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Basic Result (A. Schinzel): For fixed $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$, the non-reciprocal part of $F(x) = f(x)x^n + g(x)$ is irreducible or ± 1 for all sufficiently large n except for obvious situations.

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Theorem (K. Ford, S. Konyagin, F.): Let $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$ with

$$f(0) \neq 0,\ g(0) \neq 0, \text{ and } \gcd(f(x), g(x)) = 1.$$

Let r_1 and r_2 denote the number of non-zero terms in $f(x)$ and $g(x)$, respectively. If

$$n \geq \max\left\{2 \times 5^{2T-1}, 2\max\{\deg f, \deg g\}\left(5^{T-1} + \frac{1}{4}\right)\right\}$$

where

$$T = 2\,\|f\|^2 + 2\,\|g\|^2 + 2r_1 + 2r_2 - 7,$$

then the non-reciprocal part of $f(x)x^n + g(x)$ is irreducible or identically ± 1 except for obvious situations.

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Good News: Obvious situations don't occur!!

Not-So-Good News: Non-reciprocal part is irreducible if

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Theorem (M. Matthews Jr. & F.): Let $f(x)$ and $g(x)$ be $0, 1$-polynomials with

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Where does this result come from?

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ASSUME REDUCIBLE

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$$a(x)\tilde{a}(x)x^n + b(x)\tilde{b}(x)x^{n+\deg a-\deg b}$$
$$= f(x)\tilde{f}(x)x^n + g(x)\tilde{g}(x)x^{n+\deg f-\deg g}$$

$$\left(a(x)x^n + b(x)\right)\left(\tilde{b}(x)x^{n+\deg a-\deg b} + \tilde{a}(x)\right)$$

$$=\left(f(x)x^n + g(x)\right)\left(\tilde{g}(x)x^{n+\deg f-\deg g} + \tilde{f}(x)\right)$$

$$b(x)\tilde{a}(x) = g(x)\tilde{f}(x)$$

$$a(x)\tilde{b}(x)x^{2n+\deg a-\deg b} = f(x)\tilde{g}(x)x^{2n+\deg f-\deg g}$$

$$a(x)\tilde{a}(x)x^n + b(x)\tilde{b}(x)x^{n+\deg a-\deg b}$$

$$= f(x)\tilde{f}(x)x^n + g(x)\tilde{g}(x)x^{n+\deg f-\deg g}$$

$$\big(a(x)x^{N} + b(x)\big)\big(\tilde{b}(x)x^{N+\deg a-\deg b} + \tilde{a}(x)\big)$$

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$$\stackrel{?}{=}\left(f(x)x^{N}+g(x)\right)\left(\tilde{g}(x)x^{N+\deg f-\deg g}+\tilde{f}(x)\right)$$

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$$\begin{aligned}a(x)\tilde{a}(x)x^{n}&+b(x)\tilde{b}(x)x^{n+\deg a-\deg b}\\&=f(x)\tilde{f}(x)x^{n}+g(x)\tilde{g}(x)x^{n+\deg f-\deg g}\end{aligned}$$

$$\left(a(x)x^N + b(x)\right)\left(\tilde{b}(x)x^{N+\deg a - \deg b} + \tilde{a}(x)\right)$$

$$\stackrel{?}{=} \left(f(x)x^N + g(x)\right)\left(\tilde{g}(x)x^{N+\deg f - \deg g} + \tilde{f}(x)\right)$$

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$$a(x)\tilde{b}(x)x^{2N+\deg a - \deg b} = f(x)\tilde{g}(x)x^{2N+\deg f - \deg g}$$

$$a(x)\tilde{a}(x)x^N + b(x)\tilde{b}(x)x^{N+\deg a - \deg b}$$

$$= f(x)\tilde{f}(x)x^N + g(x)\tilde{g}(x)x^{N+\deg f - \deg g}$$

$$\left(a(x)x^N + b(x)\right)\left(\tilde{b}(x)x^{N+\deg a-\deg b} + \tilde{a}(x)\right)$$

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- $W(x)\widetilde{W}(x) = F(x)\tilde{F}(x)$
- $W(x)$ is a $0, 1$-polynomial with the same number of non-zero terms as $F(x)$

Lemma: Let $\boldsymbol{F(x)}$ be a $\boldsymbol{0,1}$-polynomial with $\boldsymbol{F(0) = 1}$. Then the non-reciprocal part of $\boldsymbol{F(x)}$ is reducible if and only if there exists $\boldsymbol{W(x)}$ satisfying:

- $\boldsymbol{W(x) \neq F(x)}$ and $\boldsymbol{W(x) \neq \widetilde{F}(x)}$
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Conclusion: The non-reciprocal part of

$$F(x) = f(x)x^N + g(x)$$

is reducible for N sufficiently large.

Theorem (K. Ford, S. Konyagin, F.): Let $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$ with

$$f(0) \neq 0,\ g(0) \neq 0, \text{ and } \gcd(f(x), g(x)) = 1.$$

Let r_1 and r_2 denote the number of non-zero terms in $f(x)$ and $g(x)$, respectively. If

$$n \geq \max\left\{2 \times 5^{2T-1}, 2\max\{\deg f, \deg g\}\left(5^{T-1} + \frac{1}{4}\right)\right\}$$

where

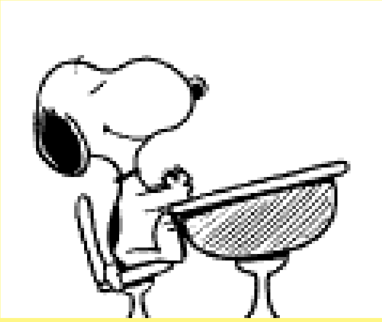
$$T = 2\,\|f\|^2 + 2\,\|g\|^2 + 2r_1 + 2r_2 - 7,$$

then the non-reciprocal part of $f(x)x^n + g(x)$ is irreducible or identically ± 1 except for obvious situations.

Contradiction

CONTRADICTION

CONTRADICTION



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