EXAMPLES OF REDUCIBLE POLYNOMIALS

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(work with Martha Allen)

Theorem (I. Schur): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

is irreducible (over the rationals) unless $n = 2^r - 1 > 1$ (when $x \pm 2$ can be a factor) or $n = 8$ (when a quadratic factor is possible).

Theorem: For n an integer ≥ 1, define

$$f(x) = \sum_{j=0}^{n} a_j \frac{x^j}{(j+1)!}$$

where the a_j's arbitrary integers with $|a_0| = 1$. Suppose

$$n + 1 = k' 2^u \quad \text{with } k' \text{ odd}$$

$$(n+1)n = k'' 2^v 3^w \quad \text{with } \gcd(k'', 6) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then $f(x)$ is irreducible.

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If

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Why is this best possible?

$$f(x) = a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

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$k,\ \ell,\ m,$ and $m' \in \mathbb{Z}^+$ (new notation)

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$$n = 2^k m \geq 3,\ n+1 = 3^\ell m',\ \gcd(mm', 6) = 1$$

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If $0 < |a_n| < mm'$, then $f(x)$ is irreducible.

$$f(x) = a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

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$$a_n = mm' \implies f(x) \text{ can have a quadratic factor}$$

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We can use Newton polygons to “guess” a quadratic factor.

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$$n = 2^k m \geq 3,\ n+1 = 3^\ell m',\ \gcd(mm', 6) = 1$$

$$a_n = mm',\ a_{n-1} = mr,\ a_{n-2} = s,$$
$$a_{n-3} = a_{n-4} = \cdots = a_3 = 0,$$
$$a_2 = -y,\ a_1 = w + y, a_0 = 1$$

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$$(n+1)! f(x) = mm' x^n + 3^\ell r mm' x^{n-1}$$
$$+ 3^\ell 2^k s mm' x^{n-2} - 3^{\ell-1} 2^{k-1} mm' (n-1)! y x^2$$

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$$\begin{aligned}(n+1)! f(x) = {} & mm'x^n + 3^\ell rmm'x^{n-1} \\ & + 3^\ell 2^k smm'x^{n-2} - 3^{\ell-1}2^{k-1}mm'(n-1)!yx^2 \\ & + 3^\ell 2^{k-1}mm'(n-1)!(w+y)x\end{aligned}$$

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$$\begin{aligned} g(x) = x^n &+ 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\ &- 3^{\ell-1} 2^{k-1} (n-1)! y x^2 \\ &+ 3^\ell 2^{k-1} (n-1)! (w+y) x \\ &+ 3^\ell 2^k (n-1)! \end{aligned}$$

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Idea: We show that there exist r, s, y, and w such that $g(x)$ is divisible by $q(x) = x^2 - 3x - 6$.

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$x^2 \equiv 3x + 6 \pmod{q(x)} \implies$

$$
\begin{aligned}
g(x) = x^n &+ 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\
&- 3^{\ell-1} 2^{k-1} (n-1)!\, y (3x+6) \\
&+ 3^\ell 2^{k-1} (n-1)!\, (w+y) x \\
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$$x^j \equiv a_j + b_j x \pmod{q(x)}$$ (new notation)

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$$\boldsymbol{a_0 = 1} \qquad\qquad\qquad \boldsymbol{b_0 = 0}$$

(2) $$\boldsymbol{a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}}$$

$$\boldsymbol{a_0 = 1 \qquad b_0 = 0}$$
$$\boldsymbol{a_1 = 0 \qquad b_1 = 1}$$

(2) $\boldsymbol{a_{j+1} = 3a_j + 6a_{j-1}, \;\; b_{j+1} = 3b_j + 6b_{j-1}}$

$$\boldsymbol{a_0 = 1} \qquad \boldsymbol{b_0 = 0}$$
$$\boldsymbol{a_1 = 0} \qquad \boldsymbol{b_1 = 1}$$
$$\boldsymbol{a_2 = 6} \qquad \boldsymbol{b_2 = 3}$$

(2) $\boldsymbol{a_{j+1} = 3a_j + 6a_{j-1}, \;\; b_{j+1} = 3b_j + 6b_{j-1}}$

$$\begin{array}{ll} \boldsymbol{a_0 = 1} & \boldsymbol{b_0 = 0} \\ \boldsymbol{a_1 = 0} & \boldsymbol{b_1 = 1} \\ \boldsymbol{a_2 = 6} & \boldsymbol{b_2 = 3} \\ \boldsymbol{a_3 = 18} & \boldsymbol{b_3 = 15} \end{array}$$

(2) $a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$

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a_0 = 1 & b_0 = 0 \\
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$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix}$$

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$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \implies A^j = \begin{pmatrix} a_j & b_j \\ a_{j+1} & b_{j+1} \end{pmatrix}$$

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$$A^j \equiv \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \pmod 4 \qquad \text{for } j > 1$$

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$$\boldsymbol{A^j \equiv \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \pmod{4}} \qquad \text{for } \boldsymbol{j > 1}$$

(3) $$\boldsymbol{\nu_2(a_j) = 1} \qquad \text{and} \qquad \boldsymbol{\nu_2(b_j) = 0}$$

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$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \implies A^j = \begin{pmatrix} a_j & b_j \\ a_{j+1} & b_{j+1} \end{pmatrix}$$

$$(4) \quad \nu_3(a_j) \geq \frac{j-1}{2} \qquad \text{and} \qquad \nu_3(b_j) \geq \frac{j-1}{2}$$

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$$\nu_3(a_{j+1}) \geq \min\{\nu_3(a_j), \nu_3(a_{j-1})\} + 1$$

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(6) $$\min\{\nu_3(a_j), \nu_3(a_{j+1})\} \leq \frac{j+1}{2}$$

$$\begin{aligned} g(x) \equiv x^n &+ 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\ &+ 3^\ell 2^{k-1}(n-1)! w x \\ &+ 3^\ell 2^k (n-1)!(1-y) \end{aligned}$$

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$$x^j \equiv a_j + b_j x \pmod{q(x)}$$

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$$

$$
\begin{aligned}
g(x) \equiv \big(b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!\big) x \\
+ a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)
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$$
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+ a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)
\end{aligned}
$$

Idea: First, we make the constant term 0. We show there exist r and s that do the trick (provided y is fixed appropriately). Note that $\gcd(3^\ell a_{n-1}, 3^\ell 2^k a_{n-2}) = 2 \times 3^v$.

First Zero Candidate:

$$a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)$$

Recall n is even and $n+1 = 3^\ell m'$.

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Recall n is even and $n+1 = 3^\ell m'$. Also,

$$\nu_3((n+1)!) < \frac{n+1}{3} + \frac{n+1}{3^2} + \cdots = \frac{n+1}{2}.$$

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Hence, $\nu_3(3^\ell 2^k (n-1)!) \leq n/2$.

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(4) $\quad \nu_3(a_j) \geq \dfrac{j-1}{2} \qquad$ and $\qquad \nu_3(b_j) \geq \dfrac{j-1}{2}$

imply $\nu_3(a_n) \geq n/2$.

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(4) $\quad \nu_3(a_j) \ge \dfrac{j-1}{2} \qquad$ and $\qquad \nu_3(b_j) \ge \dfrac{j-1}{2}$

imply $\nu_3(a_n) \ge n/2$. We fix y so that

$$\nu_3\big(a_n + 3^\ell 2^k (n-1)!(1-y)\big) \ge v.$$

It follows that there are integers r_0 and s_0 such that

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0. \tag{7}$$

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$$a_n + 3^\ell a_{n-1}\big(r_0 + 2^k a_{n-2} t\big) + 3^\ell 2^k a_{n-2}\big(s_0 - a_{n-1} t\big) + 3^\ell 2^k (n-1)!(1-y) = 0.$$

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$$r = r_0 + 2^k a_{n-2} t \qquad \text{and} \qquad s = s_0 - a_{n-1} t$$

$$
\begin{aligned}
g(x) \equiv \big(&b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!\big) x \\
&+ a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)
\end{aligned}
$$

$$\boldsymbol{g(x) \equiv \big(b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!\big) x}$$

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Next: Force the coefficient of x to be 0 by choosing w and t appropriately.

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)! \stackrel{?}{=} 0$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)! \stackrel{?}{=} 0$$

$$r = r_0 + 2^k a_{n-2} t \qquad \text{and} \qquad s = s_0 - a_{n-1} t$$

$$a_n = 3a_{n-1} + 6a_{n-2}, \quad b_n = 3b_{n-1} + 6b_{n-2}$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w(n-1)! \stackrel{?}{=} 0$$

$$r = r_0 + 2^k a_{n-2} t \qquad \text{and} \qquad s = s_0 - a_{n-1} t$$

$$a_n = 3a_{n-1} + 6a_{n-2}, \quad b_n = 3b_{n-1} + 6b_{n-2}$$

(8) $$3^\ell 2^{k-1} w(n-1)! + 3^\ell 2^k \big(a_{n-2} b_{n-1} - a_{n-1} b_{n-2}\big) t + (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w(n-1)! \stackrel{?}{=} 0$$

$$r = r_0 + 2^k a_{n-2} t \qquad \text{and} \qquad s = s_0 - a_{n-1} t$$

$$a_n = 3a_{n-1} + 6a_{n-2}, \quad b_n = 3b_{n-1} + 6b_{n-2}$$

$$(8) \quad 3^\ell 2^{k-1} w(n-1)! + 3^\ell 2^k \big(a_{n-2} b_{n-1} - a_{n-1} b_{n-2}\big) t + (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.$$

$$(9) \quad (3^\ell r_0 + 3) a_{n-1} + (3^\ell 2^k s_0 + 6) a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0.$$

$$
\begin{aligned}
(8)\qquad & 3^\ell 2^{k-1} w(n-1)! \\
& + 3^\ell 2^k \big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big)t \\
& + (3^\ell r_0 + 3)b_{n-1} + (3^\ell 2^k s_0 + 6)b_{n-2} = 0.
\end{aligned}
$$

$$
\begin{aligned}
(9)\qquad & (3^\ell r_0 + 3)a_{n-1} + (3^\ell 2^k s_0 + 6)a_{n-2} \\
& \qquad + 3^\ell 2^k (n-1)!(1-y) = 0.
\end{aligned}
$$

(8) $\times\, a_{n-1}$ − (9) $\times\, b_{n-1}$:

(8)
$$\begin{aligned}&3^\ell 2^{k-1} w(n-1)!\\&+3^\ell 2^k\big(a_{n-2}b_{n-1}-a_{n-1}b_{n-2}\big)t\\&+(3^\ell r_0+3)b_{n-1}+(3^\ell 2^k s_0+6)b_{n-2}=0.\end{aligned}$$

(9)
$$\begin{aligned}&(3^\ell r_0+3)a_{n-1}+(3^\ell 2^k s_0+6)a_{n-2}\\&\qquad+3^\ell 2^k(n-1)!(1-y)=0.\end{aligned}$$

(8) $\times\, a_{n-1}$ − (9) $\times\, b_{n-1}$:

(10)
$$\begin{aligned}&a_{n-1}3^\ell 2^{k-1}(n-1)!w\\&+a_{n-1}3^\ell 2^k\big(a_{n-2}b_{n-1}-a_{n-1}b_{n-2}\big)t\\&-(3^\ell 2^k s_0+6)\big(a_{n-2}b_{n-1}-a_{n-1}b_{n-2}\big)\\&-3^\ell 2^k(n-1)!(1-y)b_{n-1}=0.\end{aligned}$$

$$3^\ell 2^{k-1} w(n-1)! + 3^\ell 2^k \big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big)t + (3^\ell r_0 + 3)b_{n-1} + (3^\ell 2^k s_0 + 6)b_{n-2} = 0. \tag{8}$$

$$(3^\ell r_0 + 3)a_{n-1} + (3^\ell 2^k s_0 + 6)a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0. \tag{9}$$

(8) $\times\, a_{n-2}$ $-$ (9) $\times\, b_{n-2}$:

(8)
$$\begin{aligned}&3^\ell 2^{k-1} w(n-1)!\\ &\quad + 3^\ell 2^k \big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big)t\\ &\quad + (3^\ell r_0 + 3)b_{n-1} + (3^\ell 2^k s_0 + 6)b_{n-2} = 0.\end{aligned}$$

(9)
$$\begin{aligned}&(3^\ell r_0 + 3)a_{n-1} + (3^\ell 2^k s_0 + 6)a_{n-2}\\ &\qquad\qquad + 3^\ell 2^k (n-1)!(1-y) = 0.\end{aligned}$$

(8) $\times\, a_{n-2}$ − (9) $\times\, b_{n-2}$:

(11)
$$\begin{aligned}&a_{n-2} 3^\ell 2^{k-1}(n-1)! w\\ &\quad + a_{n-2} 3^\ell 2^k \big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big)t\\ &\quad + (3^\ell r_0 + 3)\big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big)\\ &\quad - 3^\ell 2^k (n-1)!(1-y)b_{n-2} = 0.\end{aligned}$$

We work with (10) if $\nu_3(a_{n-1}) \le \nu_3(a_{n-2})$ and work with (11) otherwise.

Suppose $\nu_3(a_{n-1}) \le \nu_3(a_{n-2})$. Let

$$\begin{aligned} c &= a_{n-1}3^{\ell}2^{k-1}(n-1)!, \\ c' &= a_{n-1}3^{\ell}2^{k}\big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big), \\ c'' &= (3^{\ell}2^{k}s_0 + 6)\big(a_{n-2}b_{n-1} - a_{n-1}b_{n-2}\big), \\ c''' &= 3^{\ell}2^{k}(n-1)!(1-y)b_{n-1}. \end{aligned}$$

Show there exist w and t such that $cw + c't = c'' + c'''$ by looking at the ν_2 and ν_3 values.