

# **EXAMPLES OF REDUCIBLE POLYNOMIALS**

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(work with Martha Allen)

**Theorem (I. Schur):** Let  $n$  be a positive integer, and let  $a_0, a_1, \dots, a_n$  denote arbitrary integers with  $|a_0| = |a_n| = 1$ . Then

$$a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \dots + a_1 \frac{x}{2} + a_0$$

is irreducible (over the rationals) unless  $n = 2^r - 1 > 1$  (when  $x \pm 2$  can be a factor) or  $n = 8$  (when a quadratic factor is possible).

**Theorem:** For  $n$  an integer  $\geq 1$ , define

$$f(x) = \sum_{j=0}^n a_j \frac{x^j}{(j+1)!}$$

where the  $a_j$ 's arbitrary integers with  $|a_0| = 1$ . Suppose

$$n + 1 = k'2^u \quad \text{with } k' \text{ odd}$$

$$(n + 1)n = k''2^v3^w \quad \text{with } \gcd(k'', 6) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then  $f(x)$  is irreducible.

**Theorem:** For  $n$  an integer  $\geq 1$ , define

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If

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then  $f(x)$  is irreducible.

Why is this best possible?

$$f(x) = a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

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$k, \ell, m,$  and  $m' \in \mathbb{Z}^+$  (new notation)

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$k, \ell, m,$  and  $m' \in \mathbb{Z}^+$  (new notation)

$$n = 2^k m \geq 3, \quad n + 1 = 3^\ell m', \quad \gcd(mm', 6) = 1$$

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$$n = 2^k m \geq 3, \quad n + 1 = 3^\ell m', \quad \gcd(mm', 6) = 1$$

$$a_j \in \mathbb{Z}, \quad a_0 = \pm 1$$

If  $0 < |a_n| < mm'$ , then  $f(x)$  is irreducible.

$$f(x) = a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

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$$n = 2^k m \geq 3, n+1 = 3^\ell m', \gcd(mm', 6) = 1$$

$$a_n = mm' \implies f(x) \text{ can have a quadratic factor}$$

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We can use Newton polygons to “guess” a quadratic factor.

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$$n = 2^k m \geq 3, n+1 = 3^\ell m', \gcd(mm', 6) = 1$$

$$a_n = mm', a_{n-1} = mr, a_{n-2} = s,$$

$$a_{n-3} = a_{n-4} = \cdots = a_3 = 0,$$

$$a_2 = -y, a_1 = w + y, a_0 = 1$$

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$$(n+1)!f(x) =$$

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$$(n+1)!f(x) = mm'x^n$$

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$$(n+1)!f(x) = mm'x^n + 3^\ell rmm'x^{n-1}$$

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$$(n+1)!f(x) = mm'x^n + 3^\ell rmm'x^{n-1} + 3^\ell 2^k smm'x^{n-2}$$

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$$(n+1)!f(x) = mm'x^n + 3^\ell rmm'x^{n-1} \\ + 3^\ell 2^k smm'x^{n-2} - 3^{\ell-1} 2^{k-1} mm'(n-1)!yx^2$$

$$f(x) = a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

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$$\begin{aligned}g(x) = & x^n + 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\ & - 3^{\ell-1} 2^{k-1} (n-1)! y x^2 \\ & + 3^\ell 2^{k-1} (n-1)! (w+y)x \\ & + 3^\ell 2^k (n-1)!\end{aligned}$$

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**Idea:** We show that there exist  $r$ ,  $s$ ,  $y$ , and  $w$  such that  $g(x)$  is divisible by  $q(x) = x^2 - 3x - 6$ .

$$\begin{aligned}
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$$x^2 \equiv 3x + 6 \pmod{q(x)} \implies$$

$$\begin{aligned}
g(x) = & x^n + 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\
& - 3^{\ell-1} 2^{k-1} (n-1)! y (3x + 6) \\
& + 3^\ell 2^{k-1} (n-1)! (w + y)x \\
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$$x^j \equiv a_j + b_j x \pmod{q(x)} \quad (\text{new notation})$$

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$$(1) \quad x^{j+1} \equiv 3x^j + 6x^{j-1} \pmod{q(x)}$$

$$x^j \equiv a_j + b_j x \pmod{q(x)} \quad (\text{new notation})$$

$$(1) \quad x^{j+1} \equiv 3x^j + 6x^{j-1} \pmod{q(x)}$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

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$$a_0 = 1$$

$$b_0 = 0$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

$$a_0 = 1$$

$$a_1 = 0$$

$$b_0 = 0$$

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$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

$$a_0 = 1$$

$$a_1 = 0$$

$$a_2 = 6$$

$$b_0 = 0$$

$$b_1 = 1$$

$$b_2 = 3$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

$$a_0 = 1$$

$$b_0 = 0$$

$$a_1 = 0$$

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$$a_2 = 6$$

$$b_2 = 3$$

$$a_3 = 18$$

$$b_3 = 15$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

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$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix}$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

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$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \implies A^j = \begin{pmatrix} a_j & b_j \\ a_{j+1} & b_{j+1} \end{pmatrix}$$

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$$\begin{array}{ll} a_0 = 1 & b_0 = 0 \\ a_1 = 0 & b_1 = 1 \\ a_2 = 6 & b_2 = 3 \\ a_3 = 18 & b_3 = 15 \end{array}$$

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \implies A^j = \begin{pmatrix} a_j & b_j \\ a_{j+1} & b_{j+1} \end{pmatrix}$$

$$A^j \equiv \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \pmod{4} \quad \text{for } j > 1$$

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$$A^j \equiv \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \pmod{4} \quad \text{for } j > 1$$

$$(3) \quad \nu_2(a_j) = 1 \quad \text{and} \quad \nu_2(b_j) = 0$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \implies A^j = \begin{pmatrix} a_j & b_j \\ a_{j+1} & b_{j+1} \end{pmatrix}$$

$$(4) \quad \nu_3(a_j) \geq \frac{j-1}{2} \quad \text{and} \quad \nu_3(b_j) \geq \frac{j-1}{2}$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

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$$(4) \quad \nu_3(a_j) \geq \frac{j-1}{2} \quad \text{and} \quad \nu_3(b_j) \geq \frac{j-1}{2}$$

$$\nu_3(a_{j+1}) \geq \min\{\nu_3(a_j), \nu_3(a_{j-1})\} + 1$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \implies A^j = \begin{pmatrix} a_j & b_j \\ a_{j+1} & b_{j+1} \end{pmatrix}$$

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$$(5) \quad a_j b_{j+1} - a_{j+1} b_j = \pm 6^j$$

$$(2) \quad a_{j+1} = 3a_j + 6a_{j-1}, \quad b_{j+1} = 3b_j + 6b_{j-1}$$

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$$(6) \quad \min\{\nu_3(a_j), \nu_3(a_{j+1})\} \leq \frac{j+1}{2}$$

$$\begin{aligned}g(x) \equiv & x^n + 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\ & + 3^\ell 2^{k-1} (n-1)! w x \\ & + 3^\ell 2^k (n-1)! (1-y)\end{aligned}$$

$$\begin{aligned}g(x) &\equiv x^n + 3^\ell r x^{n-1} + 3^\ell 2^k s x^{n-2} \\ &\quad + 3^\ell 2^{k-1} (n-1)! w x \\ &\quad + 3^\ell 2^k (n-1)! (1-y) \\ x^j &\equiv a_j + b_j x \pmod{q(x)}\end{aligned}$$

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\end{aligned}$$

$$x^j \equiv a_j + b_j x \pmod{q(x)}$$

$$\begin{aligned}
g(x) &\equiv (b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!) x \\
&\quad + a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)! (1-y)
\end{aligned}$$

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g(x) &\equiv (b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!) x \\
&\quad + a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)! (1-y)
\end{aligned}$$

**Idea:** First, we make the constant term 0. We show there exist  $r$  and  $s$  that do the trick (provided  $y$  is fixed appropriately). Note that  $\gcd(3^\ell a_{n-1}, 3^\ell 2^k a_{n-2}) = 2 \times 3^v$ .

## First Zero Candidate:

$$a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)$$

Recall  $n$  is even and  $n + 1 = 3^\ell m'$ .

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Recall  $n$  is even and  $n+1 = 3^\ell m'$ . Also,

$$\nu_3((n+1)!) < \frac{n+1}{3} + \frac{n+1}{3^2} + \dots = \frac{n+1}{2}.$$

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Hence,  $\nu_3(3^\ell 2^k (n-1)!) \leq n/2$ .

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$$\nu_3((n+1)!) < \frac{n+1}{3} + \frac{n+1}{3^2} + \dots = \frac{n+1}{2}.$$

Hence,  $\nu_3(3^\ell 2^k (n-1)!) \leq n/2$ . Also,  $n$  even and

$$(4) \quad \nu_3(a_j) \geq \frac{j-1}{2} \quad \text{and} \quad \nu_3(b_j) \geq \frac{j-1}{2}$$

imply  $\nu_3(a_n) \geq n/2$ .

## First Zero Candidate:

$$a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)$$

Recall  $n$  is even and  $n+1 = 3^\ell m'$ . Also,

$$\nu_3((n+1)!) < \frac{n+1}{3} + \frac{n+1}{3^2} + \dots = \frac{n+1}{2}.$$

Hence,  $\nu_3(3^\ell 2^k (n-1)!) \leq n/2$ . Also,  $n$  even and

$$(4) \quad \nu_3(a_j) \geq \frac{j-1}{2} \quad \text{and} \quad \nu_3(b_j) \geq \frac{j-1}{2}$$

imply  $\nu_3(a_n) \geq n/2$ . We fix  $y$  so that

$$\nu_3(a_n + 3^\ell 2^k (n-1)!(1-y)) \geq v.$$

It follows that there are integers  $r_0$  and  $s_0$  such that

$$(7) \quad a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} \\ + 3^\ell 2^k (n-1)!(1-y) = 0.$$

It follows that there are integers  $r_0$  and  $s_0$  such that

$$(7) \quad a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} \\ + 3^\ell 2^k (n-1)!(1-y) = 0.$$

$$a_n + 3^\ell a_{n-1} (r_0 + 2^k a_{n-2} t) \\ + 3^\ell 2^k a_{n-2} (s_0 - a_{n-1} t) \\ + 3^\ell 2^k (n-1)!(1-y) = 0.$$

It follows that there are integers  $r_0$  and  $s_0$  such that

$$(7) \quad a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} \\ + 3^\ell 2^k (n-1)! (1-y) = 0.$$

$$a_n + 3^\ell a_{n-1} (r_0 + 2^k a_{n-2} t) \\ + 3^\ell 2^k a_{n-2} (s_0 - a_{n-1} t) \\ + 3^\ell 2^k (n-1)! (1-y) = 0.$$

$$r = r_0 + 2^k a_{n-2} t \quad \text{and} \quad s = s_0 - a_{n-1} t$$

$$g(x) \equiv (b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w(n-1)!)x \\ + a_n + 3^\ell r a_{n-1} + 3^\ell 2^k s a_{n-2} + 3^\ell 2^k (n-1)!(1-y)$$

$$g(x) \equiv (b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!) x$$

$$g(x) \equiv (b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)!) x$$

**Next:** Force the coefficient of  $x$  to be 0 by choosing  $w$  and  $t$  appropriately.

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)! (1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)! \stackrel{?}{=} 0$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)! (1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)! \stackrel{?}{=} 0$$

$$r = r_0 + 2^k a_{n-2} t \quad \text{and} \quad s = s_0 - a_{n-1} t$$

$$a_n = 3a_{n-1} + 6a_{n-2}, \quad b_n = 3b_{n-1} + 6b_{n-2}$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)! (1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)! \stackrel{?}{=} 0$$

$$r = r_0 + 2^k a_{n-2} t \quad \text{and} \quad s = s_0 - a_{n-1} t$$

$$a_n = 3a_{n-1} + 6a_{n-2}, \quad b_n = 3b_{n-1} + 6b_{n-2}$$

$$(8) \quad 3^\ell 2^{k-1} w (n-1)!$$

$$+ 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t$$

$$+ (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.$$

$$a_n + 3^\ell r_0 a_{n-1} + 3^\ell 2^k s_0 a_{n-2} + 3^\ell 2^k (n-1)!(1-y) = 0$$

$$b_n + 3^\ell r b_{n-1} + 3^\ell 2^k s b_{n-2} + 3^\ell 2^{k-1} w (n-1)! \stackrel{?}{=} 0$$

$$r = r_0 + 2^k a_{n-2} t \quad \text{and} \quad s = s_0 - a_{n-1} t$$

$$a_n = 3a_{n-1} + 6a_{n-2}, \quad b_n = 3b_{n-1} + 6b_{n-2}$$

$$(8) \quad 3^\ell 2^{k-1} w (n-1)!$$

$$+ 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t$$

$$+ (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.$$

$$(9) \quad (3^\ell r_0 + 3) a_{n-1} + (3^\ell 2^k s_0 + 6) a_{n-2}$$

$$+ 3^\ell 2^k (n-1)!(1-y) = 0.$$

$$\begin{aligned}
(8) \quad & 3^\ell 2^{k-1} w(n-1)! \\
& + 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t \\
& + (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.
\end{aligned}$$

$$\begin{aligned}
(9) \quad & (3^\ell r_0 + 3) a_{n-1} + (3^\ell 2^k s_0 + 6) a_{n-2} \\
& + 3^\ell 2^k (n-1)! (1-y) = 0.
\end{aligned}$$

$$(8) \times a_{n-1} - (9) \times b_{n-1}:$$

$$\begin{aligned}
(8) \quad & 3^\ell 2^{k-1} w(n-1)! \\
& + 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t \\
& + (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.
\end{aligned}$$

$$\begin{aligned}
(9) \quad & (3^\ell r_0 + 3) a_{n-1} + (3^\ell 2^k s_0 + 6) a_{n-2} \\
& + 3^\ell 2^k (n-1)! (1-y) = 0.
\end{aligned}$$

(8)  $\times a_{n-1}$  - (9)  $\times b_{n-1}$ :

$$\begin{aligned}
(10) \quad & a_{n-1} 3^\ell 2^{k-1} (n-1)! w \\
& + a_{n-1} 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t \\
& - (3^\ell 2^k s_0 + 6) (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) \\
& - 3^\ell 2^k (n-1)! (1-y) b_{n-1} = 0.
\end{aligned}$$

$$\begin{aligned}
(8) \quad & 3^\ell 2^{k-1} w(n-1)! \\
& + 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t \\
& + (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.
\end{aligned}$$

$$\begin{aligned}
(9) \quad & (3^\ell r_0 + 3) a_{n-1} + (3^\ell 2^k s_0 + 6) a_{n-2} \\
& + 3^\ell 2^k (n-1)! (1-y) = 0.
\end{aligned}$$

$$(8) \times a_{n-2} - (9) \times b_{n-2}:$$

$$\begin{aligned}
(8) \quad & 3^\ell 2^{k-1} w(n-1)! \\
& + 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t \\
& + (3^\ell r_0 + 3) b_{n-1} + (3^\ell 2^k s_0 + 6) b_{n-2} = 0.
\end{aligned}$$

$$\begin{aligned}
(9) \quad & (3^\ell r_0 + 3) a_{n-1} + (3^\ell 2^k s_0 + 6) a_{n-2} \\
& + 3^\ell 2^k (n-1)! (1-y) = 0.
\end{aligned}$$

$$(8) \times a_{n-2} - (9) \times b_{n-2}:$$

$$\begin{aligned}
(11) \quad & a_{n-2} 3^\ell 2^{k-1} (n-1)! w \\
& + a_{n-2} 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) t \\
& + (3^\ell r_0 + 3) (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}) \\
& - 3^\ell 2^k (n-1)! (1-y) b_{n-2} = 0.
\end{aligned}$$

We work with (10) if  $\nu_3(a_{n-1}) \leq \nu_3(a_{n-2})$  and work with (11) otherwise.

Suppose  $\nu_3(a_{n-1}) \leq \nu_3(a_{n-2})$ . Let

$$c = a_{n-1} 3^\ell 2^{k-1} (n-1)!,$$

$$c' = a_{n-1} 3^\ell 2^k (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}),$$

$$c'' = (3^\ell 2^k s_0 + 6) (a_{n-2} b_{n-1} - a_{n-1} b_{n-2}),$$

$$c''' = 3^\ell 2^k (n-1)! (1-y) b_{n-1}.$$

Show there exist  $w$  and  $t$  such that  $cw + c't = c'' + c'''$  by looking at the  $\nu_2$  and  $\nu_3$  values.