Applications of Padé Approximations of $(1 - z)^k$ to Number Theory

by Michael Filaseta

University of South Carolina

General Areas of Applications:

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- irrationality measures

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- the abc-conjecture

What are the Padé approximations of $(1-z)^k$?

"Man, did you scare me! For a second there I thought you were my math teacher!"

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$$(1-z)^k \approx \frac{P(z)}{Q(z)}$$

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degree $< k$ (usually)

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$$\deg P_r = \deg Q_r = r < k, \quad \deg E_r = k - r - 1$$

Some Properties of the Polynomials:

(i) $P_r(z)$, $(-z)^k Q_r(z)$, and $z^{2r+1}E_r(z)$ satisfy

$$z(z-1)y'' + \big(2r(1-z)-(k-1)z\big)y' + r(k+r)y = 0.$$

(ii) $$Q_r(z) = \sum_{j=0}^{r} \binom{2r-j}{r}\binom{k-r+j-1}{j} z^j$$

(iii) $$Q_r(z) = \frac{(k+r)!}{(k-r-1)!\,r!\,r!}\int_0^1 (1-t)^r t^{k-r-1}(1-t+zt)^r\,\mathrm{d}t$$

(iv) $$P_r(z)Q_{r+1}(z) - Q_r(z)P_{r+1}(z) = cz^{2r+1}$$

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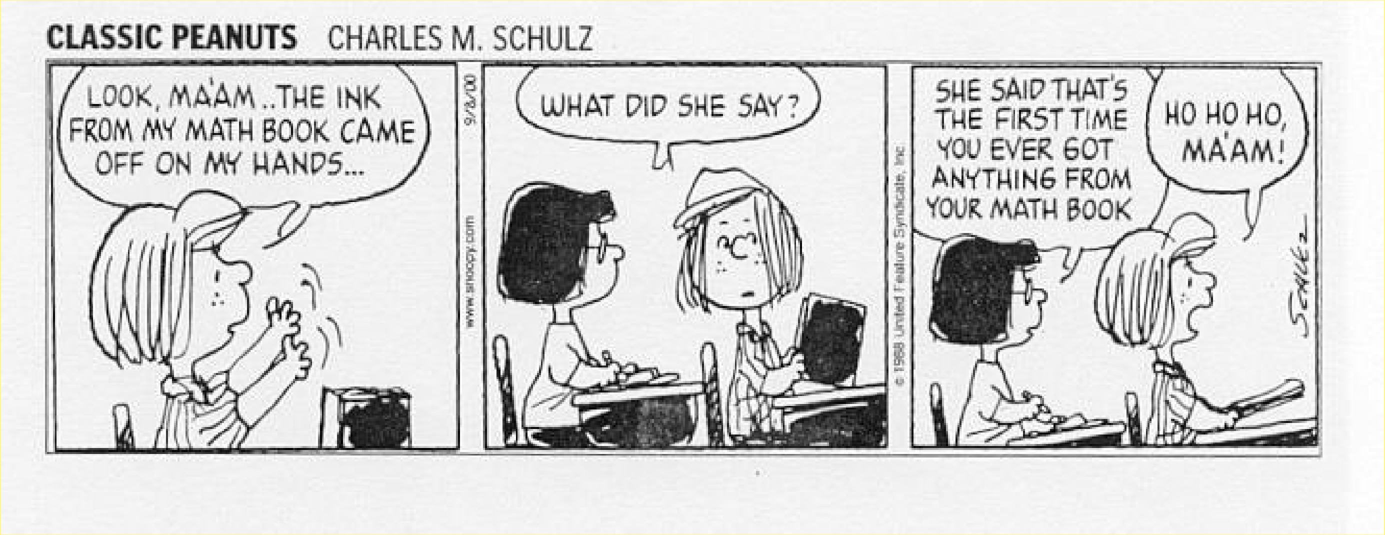
WARNING: In the applications you are about to see, the true identies used have been changed. They have been changed to conform to the identity above. The identity above gives a result of the type wanted. Typically, a closer analysis of these polynomials or even a variant of the polynomials is used to obtain the currently best known results in the applications.

Irrationality measures:



Irrationality measures:

Theorem (Liouville): Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic and of degree n. Then there is a constant $C = C(\alpha) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^n}$$

where a and b with $b > 0$ are arbitrary integers.

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Theorem (Roth): Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic. Then there is a constant $C = C(\alpha, \varepsilon) > 0$ such that

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Comment: Liouville's result is effective; Roth's is not.

Irrationality measures:

Theorem (Baker): For a and b integers with $b > 0$,

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{C}{b^{2.955}}$$

where $C = 10^{-6}$.

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Theorem (Baker): For a & b integers with $b > 0$,

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Theorem (Chudnovsky): For a & b integers with $b > 0$,

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Comment: Similar explicit estimates have also been made for certain other cube roots.

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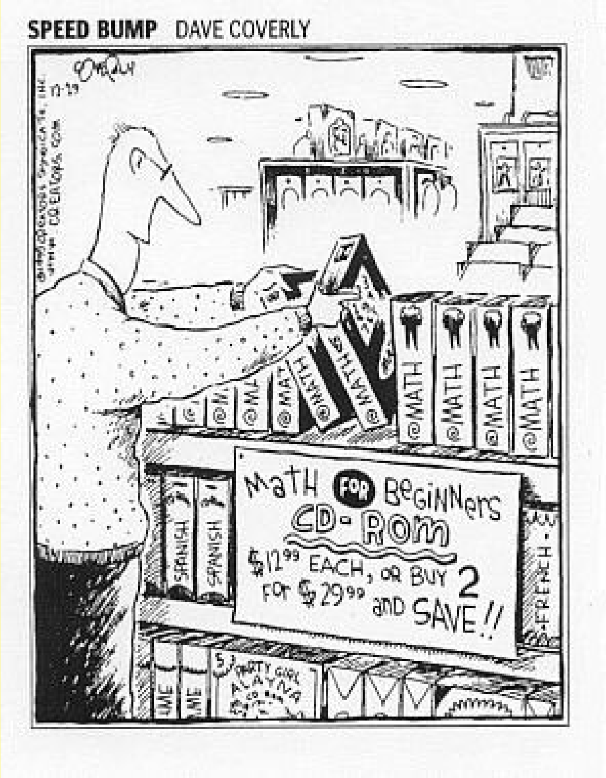
Diophantine equations:



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$$|y|^{1/2} < 4|n| \implies |y| < 16n^2$$

Diophantine equations:

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$$\left|\sqrt[3]{2} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{2\pi \mathrm{i}/3} - \frac{x}{y}\right| \left|\sqrt[3]{2}e^{4\pi \mathrm{i}/3} - \frac{x}{y}\right| = \frac{|n|}{|y|^3}$$

$$\frac{1}{4\,|y|^{2.5}} < \left|\sqrt[3]{2} - \frac{x}{y}\right| < \frac{|n|}{|y|^3}$$

$$|y|^{1/2} < 4|n| \implies |y| < 16n^2$$

Diophantine equations:

Theorem: Let n be a non-zero integer. If x and y are integers satisfying $x^3 - 2y^3 = n$, then $|y| < 16n^2$.

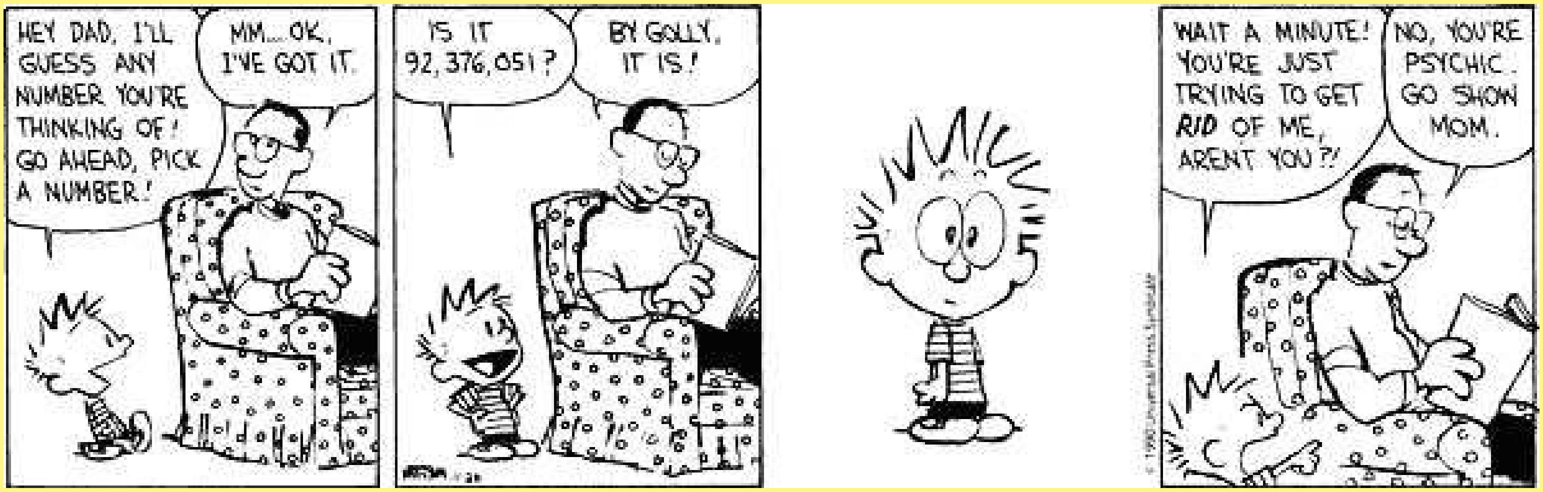
Diophantine equations:

Theorem (Bennett): If a, b, and n are integers with $ab \neq 0$ and $n \geq 3$, then the equation

$$|ax^n + by^n| = 1$$

has at most 1 solution in positive integers x and y.

Waring’s Problem:



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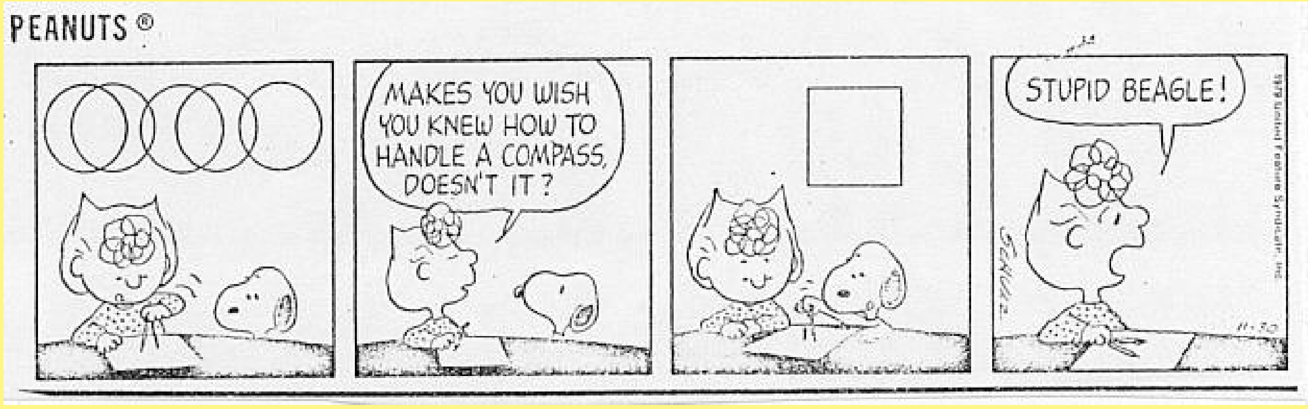
The factorization of $n(n+1)$:



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Let $p_1, p_2, \ldots, p_r$ be primes. There is an N such that if $n \geq N$ and

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Problem: Can we narrow the gap between these ineffective and effective results?

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Obtain an upper bound on 3^k. Since $3^k m_1 \geq n$, it follows that m_1 and $m = m_1 m_2$ are not small.

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More precisely, one takes $z = 1/9$ in the equation

$$P_r(z) - (1-z)^k Q_r(z) = z^{2r+1} E_r(z).$$

What’s Needed for the Method to Work:

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One largely needs to be dealing with two primes (like 2 and 3) with a difference of powers of these primes being small (like $3^2 - 2^3 = 1$).

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- D. Hilbert (1892) used his now classical Hilbert's Irreducibility Theorem to show that for each integer $n \geq 1$, there is polynomial $f(x) \in \mathbb{Z}[x]$ such that the Galois group associated with $f(x)$ is the symmetric group S_n.

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- Hilbert's work and work of E. Noether (1918) began what is now called Inverse Galois Theory.
- Van der Waerden showed that for "almost all" polynomials $f(x) \in \mathbb{Z}[x]$, the Galois group associated with $f(x)$ is the symmetric group S_n.

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- Schur did not find an explicit sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod{4}$.

Galois groups of classical polynomials:

Theorem (R. Gow, 1989): If $n > 2$ is even and

$$L_n^{(n)}(x) = \sum_{j=0}^{n} \binom{2n}{n-j} \frac{(-x)^j}{j!}$$

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Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible (and, hence, has Galois group A_n for almost all even n).

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The Ramanujan-Nagell equation:

Some Background: Beukers used a method "similar" to the approach for finding irrationality measures to show that $\sqrt{2}$ cannot be approximated too well by rationals a/b with b a power of 2. This implies bounds for solutions to the Diophantine equation $x^2 + D = 2^n$ with D fixed. He showed that if $D \neq 7$, then the equation has ≤ 4 solutions. Related work by Apéry, Beukers, and Bennett establishes that for odd primes p not dividing D, the equation $x^2 + D = p^n$ has at most 3 solutions. All of these are in some sense best possible (though more can and has been said).

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Connection with $n(n+1)$ problem:

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$$\underbrace{\left(\frac{x+\sqrt{-7}}{2}\right)}_{\text{linear}}\underbrace{\left(\frac{x-\sqrt{-7}}{2}\right)}_{\text{linear}} = \underbrace{\left(\frac{1+\sqrt{-7}}{2}\right)}_{\text{prime}}^{n-2}\underbrace{\left(\frac{1-\sqrt{-7}}{2}\right)}_{\text{prime}}^{n-2} m$$

Theorem (Bennett, F., Trifonov): If x, n and m are positive integers satisfying

$$x^2 + 7 = 2^n m \quad \text{and} \quad x \notin \{1, 3, 5, 11, 181\},$$

then

$$m \geq \text{???}$$

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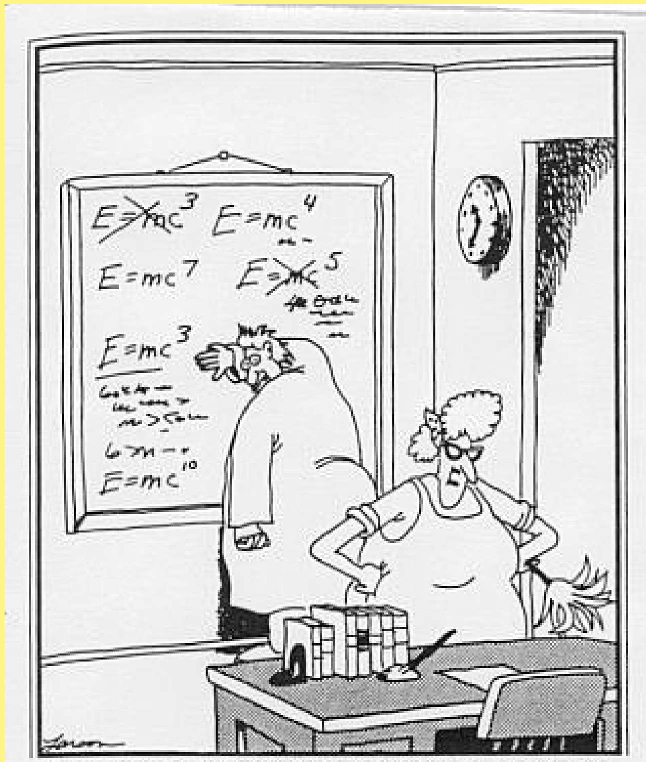
Comment: In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the 13^{th} power has absolute value ≈ 2.65 and the powers themselves have absolute value ≈ 90.51.

Intermission

k-free numbers in short intervals:



"*Now* that desk looks better. Everything's squared away, yessir, squaaaaaared away."

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Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

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Main Idea: Show there are integers in $(x, x + x^{\theta}]$ not divisible by the k^{th} power of a prime. Consider primes in different size ranges. Deal with small primes and large primes separately.

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The number of integers $n \in (x, x + x^{\theta}]$ divisible by such a p^k is bounded by $(2/3)x^{\theta}$.

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LHS small compared to RHS

$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

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where $r = \sqrt{2n} - \frac{1}{2}$.

Basic Idea: One works in a number field where $f(x)$ has a linear factor. As in the case $f(x) = x$, one wants to show certain u (in the ring of algebraic integers in the field) are not close by considering

$$(u + a)^k P - u^k Q$$

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Solution: If it's small, work with a conjugate instead.

Comment: In the case that $k \leq n$, one can *try* the same methods. The gap size becomes "bad" in the sense that one obtains $m \in (x, x+h]$ where $f(m)$ is k-free but h increases as k decreases. There is a point where h exceeds x itself and the method fails (the size of $f(m)$ is no longer of order x^n). Nair took the limit of what can be done with $k \leq n$ and obtained

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Theorem (Nair): If $f(x)$ is irreducible of degree n and $k \geq \left(2\sqrt{2}-1\right)n/2$, then there are infinitely many integers m for which $f(m)$ is k-free.

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Theorem: If $f(x,y)$ is an irreducible binary form of degree n and $k \geq (2\sqrt{2}-1)n/4$, then there are infinitely many integer pairs (a,b) for which $f(a,b)$ is k-free.

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(work of Browkin, Greaves, F., Nitaj, Schinzel)

Approach: Makes use of a preliminary result about squarefree values of binary forms.

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$$f(x,y) = xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2)$$
$$\times\,(x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)$$

the number $f(x,y)/6$ takes on the right proportion of squarefree values for

$$X < x \leq 2X, \quad Y < y \leq 2Y, \quad X = Y^{\alpha},$$

where $\alpha \in (1,3)$.

Polynomial Identity:

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$$P_3(z) - (1 - z)^7 Q_3(z) = z^7 E_3(z)$$

where

$$P_3(z) = (2z - 1)(3z^2 - 3z + 1),$$
$$Q_3(z) = -(z + 1)(z^2 + z + 1),$$

and

$$E_3(z) = -(z - 2)(z^2 - 3z + 3)$$

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$$z = \frac{x}{x+y} \Longrightarrow \begin{cases} (x+y)^7(x-y)(x^2-xy+y^2) \\ \quad + y^7(2x+y)(3x^2+3xy+y^2) \\ = x^7(x+2y)(x^2+3xy+3y^2) \end{cases}$$

$$\begin{aligned}(x+y)^7(x-y)(x^2-xy+y^2)\\ +y^7(2x+y)(3x^2+3xy+y^2)\\ =x^7(x+2y)(x^2+3xy+3y^2)\end{aligned}$$

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\end{aligned}
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&(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4) \\
&\qquad + y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4) \\
&= x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)
\end{aligned}
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\begin{aligned}
&(x+y)^7(x-y)(x^2-xy+y^2)\\
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\begin{aligned}
&(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)\\
&\qquad +y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)\\
&= x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2)\\
&\quad \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)
\end{aligned}
$$

$$\begin{aligned}(x+y)^7(x-y)(x^2-xy+y^2)\\ +y^7(2x+y)(3x^2+3xy+y^2)\\ =x^7(x+2y)(x^2+3xy+3y^2)\end{aligned}$$

$$\begin{aligned}(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)\\ +y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)\\ =x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)\end{aligned}$$

$$\begin{aligned}f(x,y)=xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2)\\ \times(x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)\end{aligned}$$

$$\begin{aligned}(x+y)^7(x-y)(x^2-xy+y^2)&\\ +y^7(2x+y)(3x^2+3xy+y^2)&\\ =x^7(x+2y)(x^2+3xy+3y^2)&\end{aligned}$$

$$\begin{aligned}(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)&\\ +y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)&\\ =x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4)&\end{aligned}$$

$$\begin{aligned}f(x,y)&=xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2)\\ &\quad\times(x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)\end{aligned}$$

$$a = (x^2 + y^2)^7 (x^2 - y^2)(x^4 - x^2 y^2 + y^4)$$
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$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

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$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha \log Y}{}$$

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$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)} \approx \frac{20\alpha \log Y}{(21\alpha + 1)\log Y}$$

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$$\begin{aligned} f(x, y) = {} & xy(x + y)(x - y)(x^2 + y^2)(2x^2 + y^2)(x^2 + 2y^2) \\ & \times (x^4 - x^2 y^2 + y^4)(3x^4 + 3x^2 y^2 + y^4)(x^4 + 3x^2 y^2 + 3y^4) \end{aligned}$$

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$$1 < \alpha < 3 \implies \frac{10}{11} < L_{a,b} < \frac{15}{16}$$

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Comment: This shows $[10/11, 15/16]$ is contained in the set of limit points of $L_{a,b}$. A similar argument is given for other subintervals of $[1/3, 36/37]$ (not all involving Padé approximations).