APPLICATIONS OF PADÉ APPROXIMATIONS OF $(1-z)^k$ to Number Theory

by Michael Filaseta University of South Carolina

• irrationality measures

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- diophantine equations

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- Waring's problem

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- *k*-free numbers in short intervals
- k-free values of polynomials and binary forms
- the *abc*-conjecture

Answer: Rational functions that give good approximations to $(1-z)^k$ near the origin.

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What are the Padé approximations of e^{z} ?

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Important Equation:

$$P_r - (1-z)^k Q_r = z^{2r+1} E_r$$

 $\deg P_r = \deg Q_r = r < k, \quad \deg E_r = k - r - 1$

Some Properties of the Polynomials:

(i)
$$P_r(z), (-z)^k Q_r(z)$$
, and $z^{2r+1} E_r(z)$ satisfy
 $z(z-1)y'' + (2r(1-z)-(k-1)z)y' + r(k+r)y = 0.$

(ii)
$$Q_r(z) = \sum_{j=0}^r {\binom{2r-j}{r} \binom{k-r+j-1}{j} z^j}$$

(iii)
$$Q_r(x) = \frac{(k+r)!}{(k-r-1)! r! r!} \int_0^1 (1-t)^r t^{k-r-1} (1-t+xt)^r dt$$

(iv) $P_r(x)Q_{r+1}(x) - Q_r(x)P_{r+1}(x) = cx^{2r+1}$

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WARNING: In the applications you are about to see, this identity is used to get a result of the type wanted. Typically, a closer analysis of these polynomials or even a variant of the polynomials is needed to obtain the currently best known results in these applications.

Theorem (Liouville): Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic and of degree n. Then there is a constant $C = C(\alpha) > 0$ such that

$$\left| lpha - rac{a}{b}
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where a and b with b > 0 are arbitrary integers.

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Theorem (Roth): Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic. Then there is a constant $C = C(\alpha, \varepsilon) > 0$ such that

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Comment: Liouville's result is effective; Roth's is not.

Theorem (Baker): For a and b integers with b > 0,

$$\left|\sqrt[3]{2}-rac{a}{b}
ight|>rac{C}{b^{2.955}}$$

where $C = 10^{-6}$.

Theorem (Baker): For a and b integers with b > 0, $\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{10^6 b^{2.955}}$.

Theorem (Chudnovsky): For a and b integers with b > 0,

$$\left|\sqrt[3]{2}-\frac{a}{b}\right| > \frac{1}{c \cdot b^{2.43}}.$$

Theorem (**Bennett**): For a and b integers with b > 0,

$$\left|\sqrt[3]{2}-\frac{a}{b}\right| > \frac{1}{c \cdot b^{2.47}}.$$

Irrationality measures:

Theorem (**Bennett**): For a and b integers with b > 0,

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Comment: Similar explicit estimates have also been made for certain other cube roots.

$$P_r - (1-z)^k Q_r = z^{2r+1}E_r$$

$$P_r - (1-z)^{1/3}Q_r = z^{2r+1}E_r$$

$$P_r - (\begin{array}{cc} 1-z \end{array})^{1/3} Q_r = z^{2r+1} E_r \ \uparrow \ 3/128$$

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Wait!! I thought we wanted that LARGE!!

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What's small_r?

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small
$$r < rac{1}{2b b_r}$$
 and $b_r < cb^{1.47}$.

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What's small_r? Let b be a positive integer. By choosing r right, one can obtain

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Theorem: Let n be a non-zero integer. If x and y are integers satisfying $x^3 - 2y^3 = n$, then $|y| < 16n^2$.

Theorem (Bennett): If a, b, and n are integers with $ab \neq 0$ and $n \geq 3$, then the equation

 $|ax^n + by^n| = 1$

has at most one solution in positive integers x and y.

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Known: (i)
$$g(k) = 2^k + \left[\left(\frac{3}{2} \right)^k \right] - 2$$

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(iii) (i) holds if $\left\| \left(\frac{3}{2}\right)^k \right\| > 0.75^k$

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Theorem (Beukers): If k > 4, then $\left\| \left(\frac{3}{2}\right)^k \right\| > 0.5358^k.$

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Theorem (Dubitskas): If k > 4, then $\left\| \left(\frac{3}{2}\right)^k \right\| > 0.5767^k.$

The factorization of n(n + 1):

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Let p_1, p_2, \ldots, p_r be primes. There is an N such that if $n \geq N$ and

$$n(n+1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

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for some integer m, then m > 1.

Lehmer: Gave some explicit estimates:

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Problem: Can we narrow the gap between these ineffective and effective results?

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Conjecture: For n > 512,

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Obtain an upper bound on 3^k . Since $3^k m_1 \ge n$, it follows that m_1 and, hence, $m = m_1 m_2$ are not small.

The "small" integers P, Q, and E are obtained through the use of Padé approximations for $(1 - x)^k$.

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More precisely, one takes z = 1/9 in the equation

$$P_r(x) - (1-x)^k Q_r(x) = x^{2r+1} E_r(x).$$

What's Needed for the Method to Work:

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One largely needs to be dealing with two primes (like 2 and 3) with a difference of powers of these primes being small (like $3^2 - 2^3 = 1$).

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- Hilbert's work and work of E. Noether (1918) began what has come to be known as Inverse Galois Theory.
- Van der Waerden showed that for "almost all" polynomials $f(x) \in \mathbb{Z}[x]$, the Galois group associated with f(x) is the symmetric group S_n .

• Schur showed $L_n^{(0)}(x)$ has Galois group S_n .

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• Schur did not find an explicit sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod{4}$.

Theorem (R. Gow, 1989): If n > 2 is even and

$$L_n^{(n)}(x)=\sum_{j=0}^n {2n \choose n-j}rac{(-x)^j}{j!}$$

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Work in Progress with Trifonov: There is an effetive bound N such that if $n \ge N$ and $n \equiv 2 \pmod{4}$, then $L_n^{(n)}(x)$ is irreducible.

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Classical Ramanujan-Nagell Theorem: If x and n are integers satisfying

$$x^2 + 7 = 2^n,$$

then

$$x \in \{1, 3, 5, 11, 181\}.$$

Some Background: Beukers used a method "similar" to the approach for finding irrationality measures to show that $\sqrt{2}$ cannot be approximated too well by rationals a/bwith b a power of 2. This implies bounds for solutions to the Diophantine equation $x^2 + D = 2^n$ with D fixed. This led to him showing that if $D \neq 7$, then the equation has at most 4 solutions. Related independent work by Apéry, Beukers, and Bennett establishes that for odd primes p not dividing D, the equation $x^2 + D = p^n$ has at most 3 solutions. All of these are in some sense best possible (though more can and has been said).

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Problem: If $x^2 + 7 = 2^n m$ and x is not in the set above, then can we say that m must be large?

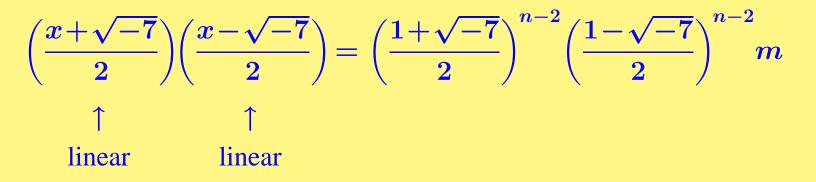
Connection with n(n + 1) **problem:**

 $x^2 + 7 = 2^n m$

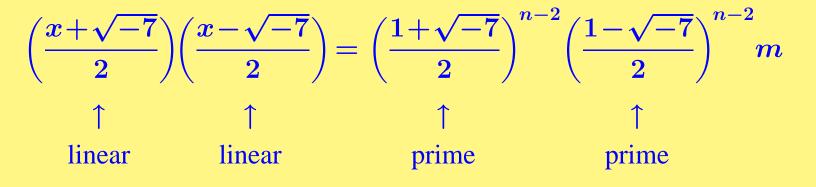
$$x^2 + 7 = 2^n m$$

$$igg(rac{x\!+\!\sqrt{-7}}{2}igg)\!igg(rac{x\!-\!\sqrt{-7}}{2}igg) = igg(rac{1\!+\!\sqrt{-7}}{2}igg)^{n-2}\!igg(rac{1\!-\!\sqrt{-7}}{2}igg)^{n-2}m$$

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Theorem (Bennett, F., Trifonov): If *x*, *n* and *m* are positive integers satisfying

 $x^2 + 7 = 2^n m$ and $x \not\in \{1, 3, 5, 11, 181\},$ then

$$m \geq ???$$

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Comment: In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the 13th power has absolute value ≈ 2.65 and the powers themselves have absolute value ≈ 90.51 .

k-free numbers in short intervals:

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Problem: Find $\theta = \theta(k)$ as small as possible such that, for x sufficiently large, the interval $(x, x + x^{\theta}]$ contains a k-free number.

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Main Idea: Show that there are integers in $(x, x + x^{\theta}]$ not divisible by the k^{th} power of a prime. Consider primes in different size ranges. Deal with small primes and large primes separately.

Small Primes: $p \leq z$

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$$\sum_{p \leq z} \left(\frac{x^{\theta}}{p^k} + 1 \right) \leq \left(\sum_{p \text{ prime}} \frac{x^{\theta}}{p^2} \right) + \pi(z)$$

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$$egin{split} \sum\limits_{p\leq z}\left(rac{x^{ heta}}{p^k}+1
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ight)+\pi(z)\ &\leq \Big(rac{\pi^2}{6}-1\Big)x^{ heta} \end{split}$$

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Idea: Show that there are few primes $p \in (N, 2N]$ with x/p^k that close to an integer.

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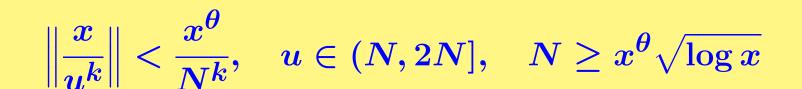
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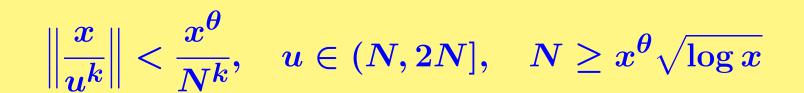
Exponential Sums:

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Exponential Sums: Let $\delta \in (0, 1/2)$. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Let S be a set of positive integers. Then for any positive integer $J \leq 1/(4\delta)$, we get

$$egin{aligned} &|\{u\in S: \|f(u)\|<\delta\}|\ &\leq rac{\pi^2}{2(J+1)}\sum_{1\leq j\leq J}\Big|\sum_{u\in S}e^{2\pi\mathrm{i} jf(u)}\Big|\ &+rac{\pi^2}{4(J+1)}|S|. \end{aligned}$$





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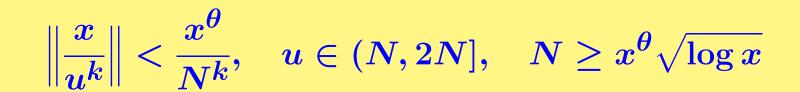
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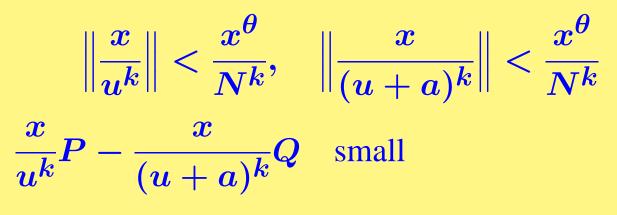
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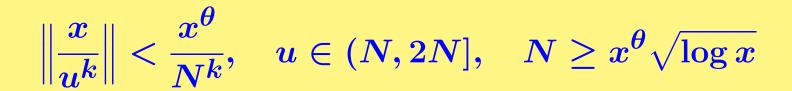
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$\ u^k \ \ge N^{k^{\gamma}}$		

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 $s \in \mathbb{Q} - \{-(k-1), -(k-2), \dots, k-2, k-1\}.$ Let $f(u) = X/u^s$. Suppose that

 $N^s \leq X$ and $\delta \leq c N^{-(k-1)}$,

where c > 0 is small. Set

 $S = \{u \in \mathbb{Z} \cap (N, 2N] : \|f(u)\| < \delta\}.$

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k-free values of polynomials and binary forms:

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Basic Idea: One works in a number field where f(x) has a linear factor. As in the case f(x) = x, one wants to show certain u (in the ring of algebraic integers in the field) are not close by considering

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Solution: If it's small, work with a conjugate instead.

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- **Theorem (Nair):** If f(x) is an irreducible polynomial of degree n and $k \ge (2\sqrt{2} 1)n/2$, then there are infinitely many integers m for which f(m) is k-free.
- **Theorem:** If f(x, y) is an irreducible binary form of degree n and $k \ge (2\sqrt{2} 1)n/4$, then there are infinitely many integer pairs (a, b) for which f(a, b) is k-free.

The *abc*-conjecture:

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The *abc*-Conjecture: For *a* and *b* in \mathbb{Z}^+ , define $L_{a,b} = \frac{\log(a+b)}{\log Q(ab(a+b))}$

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 $\mathcal{L} = \{L_{a,b} : a \ge 1, b \ge 1, \gcd(a, b) = 1\}.$ The set of limit points of \mathcal{L} is the interval [1/3, 1].

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(work of Browkin, Greaves, F., Nitaj, Schinzel)

Approach: Makes use of a preliminary result about square-free values of binary forms.

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the number f(x, y)/6 takes on the right proportion of squarefree values for

 $X < x \leq 2X, \hspace{1em} Y < y \leq 2Y, \hspace{1em} X = Y^lpha,$ where $lpha \in (1,3).$

$$P_3(z) - (1-z)^7 Q_3(z) = z^7 E_3(z)$$

where

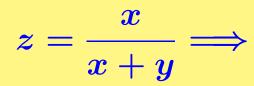
$$egin{aligned} P_3(z) &= (2z-1)(3z^2-3z+1), \ Q_3(z) &= -(z+1)(z^2+z+1), \end{aligned}$$

and

$$E_3(z) = -(z-2)(z^2-3z+3)$$

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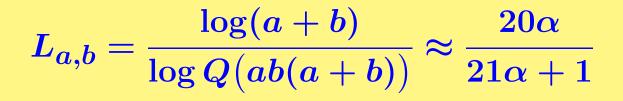
$$egin{aligned} f(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2)\ & imes (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4) \end{aligned}$$

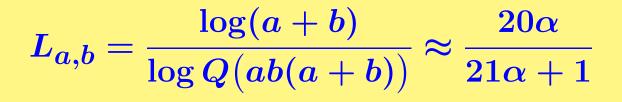
$$L_{a,b} = rac{\log(a+b)}{\log Q(ab(a+b))} pprox rac{20lpha \log Y}{(21lpha+1)\log Y}$$

$$egin{aligned} a &= (x^2+y^2)^7 (x^2-y^2) (x^4-x^2y^2+y^4) \ b &= y^{14} (2x^2+y^2) (3x^4+3x^2y^2+y^4) \ X &= Y^lpha, \ 1 < lpha < 3 \ a+b &= x^{14} (x^2+2y^2) (x^4+3x^2y^2+3y^4) \end{aligned}$$

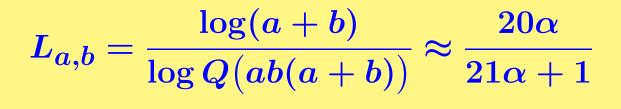
$$egin{aligned} f(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \ & imes (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4) \end{aligned}$$

$$L_{a,b} = rac{\log(a+b)}{\log Q(ab(a+b))} pprox rac{20lpha}{21lpha+1}$$

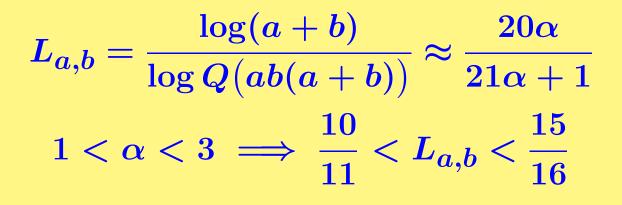




 $1 < \alpha < 3 \implies$



 $1 < \alpha < 3 \implies ?? < L_{a,b} < ??$



$$egin{aligned} L_{a,b} &= rac{\log(a+b)}{\log Qig(ab(a+b)ig)} pprox rac{20lpha}{21lpha+1} \ 1 &< lpha < 3 \implies rac{10}{11} < L_{a,b} < rac{15}{16} \end{aligned}$$

Comment: This shows [10/11, 15/16] is contained in the set of limit points of $L_{a,b}$. A similar argument is given for other subintervals of [1/3, 36/37] (not all involving Padé approximations).