Applications of Padé Approximations of $(1-z)^k$ to Number Theory

by Michael Filaseta

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General Areas of Applications:

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- irrationality measures

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- the abc-conjecture

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$$\deg P_r = \deg Q_r = r < k, \quad \deg E_r = k - r - 1$$

Some Properties of the Polynomials:

(i) $P_r(z)$, $(-z)^k Q_r(z)$, and $z^{2r+1}E_r(z)$ satisfy

$$z(z-1)y'' + \big(2r(1-z)-(k-1)z\big)y' + r(k+r)y = 0.$$

(ii) $$Q_r(z) = \sum_{j=0}^{r} \binom{2r-j}{r}\binom{k-r+j-1}{j} z^j$$

(iii) $$Q_r(x) = \frac{(k+r)!}{(k-r-1)!\,r!\,r!}\int_0^1 (1-t)^r t^{k-r-1}(1-t+xt)^r\,\mathrm{d}t$$

(iv) $$P_r(x)Q_{r+1}(x) - Q_r(x)P_{r+1}(x) = cx^{2r+1}$$

$$\boldsymbol{P_r - (1 - z)^k Q_r = z^{2r+1} E_r}$$

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WARNING: In the applications you are about to see, this identity is used to get a result of the type wanted. Typically, a closer analysis of these polynomials or even a variant of the polynomials is needed to obtain the currently best known results in these applications.

Irrationality measures:

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Theorem (Liouville): Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic and of degree n. Then there is a constant $C = C(\alpha) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{b^n}$$

where a and b with $b > 0$ are arbitrary integers.

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Theorem (Roth): Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with α algebraic. Then there is a constant $C = C(\alpha, \varepsilon) > 0$ such that

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Comment: Liouville's result is effective; Roth's is not.

Irrationality measures:

Theorem (Baker): For a and b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{C}{b^{2.955}}$$

where $C = 10^{-6}$.

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Theorem (Chudnovsky): For a and b integers with $b > 0$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{c \cdot b^{2.43}}.$$

Irrationality measures:

Theorem (Bennett): For $\boldsymbol{a}$ and $\boldsymbol{b}$ integers with $\boldsymbol{b > 0}$,

$$\left|\sqrt[3]{2} - \frac{a}{b}\right| > \frac{1}{c \cdot b^{2.47}}.$$

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Theorem (Bennett): For a and b integers with $b > 0$,

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Comment: Similar explicit estimates have also been made for certain other cube roots.

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What's small$_r$? Let b be a positive integer. By choosing r right, one can obtain

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Diophantine equations:

Theorem: Let n be a non-zero integer. If x and y are integers satisfying $x^3 - 2y^3 = n$, then $|y| < 16n^2$.

Diophantine equations:

Theorem (Bennett): If a, b, and n are integers with $ab \neq 0$ and $n \geq 3$, then the equation

$$|ax^n + by^n| = 1$$

has at most one solution in positive integers x and y.

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Known: (i) $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$

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Theorem (Beukers): If $\boldsymbol{k} > 4$, then

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Problem: Can we narrow the gap between these ineffective and effective results?

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Obtain an upper bound on 3^k. Since $3^k m_1 \geq n$, it follows that m_1 and, hence, $m = m_1 m_2$ are not small.

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More precisely, one takes $z = 1/9$ in the equation

$$P_r(x) - (1-x)^k Q_r(x) = x^{2r+1} E_r(x).$$

What’s Needed for the Method to Work:

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One largely needs to be dealing with two primes (like 2 and 3) with a difference of powers of these primes being small (like $\mathbf{3^2 - 2^3 = 1}$).

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- D. Hilbert (1892) used his now classical Hilbert's Irreducibility Theorem to show that for each integer $n \geq 1$, there is polynomial $f(x) \in \mathbb{Z}[x]$ such that the Galois group associated with $f(x)$ is the symmetric group S_n.

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- Van der Waerden showed that for "almost all" polynomials $f(x) \in \mathbb{Z}[x]$, the Galois group associated with $f(x)$ is the symmetric group S_n.

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- Schur did not find an explicit sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod 4$.

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Theorem (R. Gow, 1989): If $n > 2$ is even and

$$L_n^{(n)}(x) = \sum_{j=0}^{n} \binom{2n}{n-j} \frac{(-x)^j}{j!}$$

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Classical Ramanujan-Nagell Theorem: If x and n are integers satisfying

$$x^2 + 7 = 2^n,$$

then

$$x \in \{1, 3, 5, 11, 181\}.$$

The Ramanujan-Nagell equation:

Some Background: Beukers used a method "similar" to the approach for finding irrationality measures to show that $\sqrt{2}$ cannot be approximated too well by rationals a/b with b a power of 2. This implies bounds for solutions to the Diophantine equation $x^2 + D = 2^n$ with D fixed. This led to him showing that if $D \neq 7$, then the equation has at most 4 solutions. Related independent work by Apéry, Beukers, and Bennett establishes that for odd primes p not dividing D, the equation $x^2 + D = p^n$ has at most 3 solutions. All of these are in some sense best possible (though more can and has been said).

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Comment: In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the 13^{th} power has absolute value ≈ 2.65 and the powers themselves have absolute value ≈ 90.51.

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Main Idea: Show that there are integers in $(x, x + x^{\theta}]$ not divisible by the k^{th} power of a prime. Consider primes in different size ranges. Deal with small primes and large primes separately.

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Exponential Sums: Let $\delta \in (0, 1/2)$. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Let S be a set of positive integers. Then for any positive integer $J \leq 1/(4\delta)$, we get

$$\begin{aligned} |\{u \in S : \|f(u)\| < \delta\}| \\ &\leq \frac{\pi^2}{2(J+1)} \sum_{1 \leq j \leq J} \Big| \sum_{u \in S} e^{2\pi \mathrm{i} j f(u)} \Big| \\ &\quad + \frac{\pi^2}{4(J+1)} |S|. \end{aligned}$$

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LHS small compared to RHS

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Theorem (Trifonov): For x sufficiently large, there is a k-free number in $(x, x + cx^{1/(2k+1)} \log x]$.

More General Theorem (F. & Trifonov): Let k be an integer ≥ 2, and let

$$s \in \mathbb{Q} - \{-(k-1), -(k-2), \ldots, k-2, k-1\}.$$

Let $f(u) = X/u^s$. Suppose that

$$N^s \leq X \quad \text{and} \quad \delta \leq cN^{-(k-1)},$$

where $c > 0$ is small. Set

$$S = \{u \in \mathbb{Z} \cap (N, 2N] : \|f(u)\| < \delta\}.$$

Then

$$|S| \ll_{k,s} X^{1/(2k+1)} N^{(k-s)/(2k+1)} + \delta X^{1/(6k+3)} N^{(6k^2+2k-s-1)/(6k+3)}.$$

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Theorem (Nair): Let $k \geq n+1$. For x sufficiently large, there is an integer m such that $f(m)$ is k-free with

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Basic Idea: One works in a number field where $f(x)$ has a linear factor. As in the case $f(x) = x$, one wants to show certain u (in the ring of algebraic integers in the field) are not close by considering

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Solution: If it's small, work with a conjugate instead.

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Theorem: If $f(x, y)$ is an irreducible binary form of degree n and $k \geq (2\sqrt{2}-1)n/4$, then there are infinitely many integer pairs (a, b) for which $f(a, b)$ is k-free.

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The abc-Conjecture: For a and b in $\mathbb{Z}^+$, define

$$L_{a,b} = \frac{\log(a+b)}{\log Q\big(ab(a+b)\big)}$$

and

$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}.$$

The set of limit points of $\mathcal{L}$ is the interval $[1/3, 1]$.

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$$\mathcal{L} = \{L_{a,b} : a \geq 1, b \geq 1, \gcd(a,b) = 1\}$$

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Theorem: The set of limit points of $\mathcal{L}$ includes the interval $[1/3, 36/37]$.

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(work of Browkin, Greaves, F., Nitaj, Schinzel)

Approach: Makes use of a preliminary result about square-free values of binary forms.

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$$f(x,y) = xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2) \times (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4)$$

the number $f(x,y)/6$ takes on the right proportion of squarefree values for

$$X < x \leq 2X, \quad Y < y \leq 2Y, \quad X = Y^\alpha,$$

where $\alpha \in (1,3)$.

Polynomial Identity:

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$$P_3(z) - (1-z)^7 Q_3(z) = z^7 E_3(z)$$

where

$$P_3(z) = (2z-1)(3z^2-3z+1),$$
$$Q_3(z) = -(z+1)(z^2+z+1),$$

and

$$E_3(z) = -(z-2)(z^2-3z+3)$$

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Comment: This shows $[10/11, 15/16]$ is contained in the set of limit points of $L_{a,b}$. A similar argument is given for other subintervals of $[1/3, 36/37]$ (not all involving Padé approximations).