IRREDUCIBILITY OF CLASSICAL POLYNOMIALS
AND
THEIR GENERALIZATIONS

Irreducibility of Classical Polynomials and their Generalizations

by Michael Filaseta

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- $f(x)$ does not factor as a product of two polynomials in $\mathbb{Q}[x]$ each of degree ≥ 1.

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Result (due to A. Cohn): If $f(x)$ has digits as coefficients and $f(10)$ is prime, then $f(x)$ is irreducible.

Theorem: The analogous result holds in any base $b \geq 2$.

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Comment: There exist polynomials $f(x) \in \mathbb{Z}[x]$ with non-negative coefficients with $f(10)$ prime and with $f(x)$ reducible.

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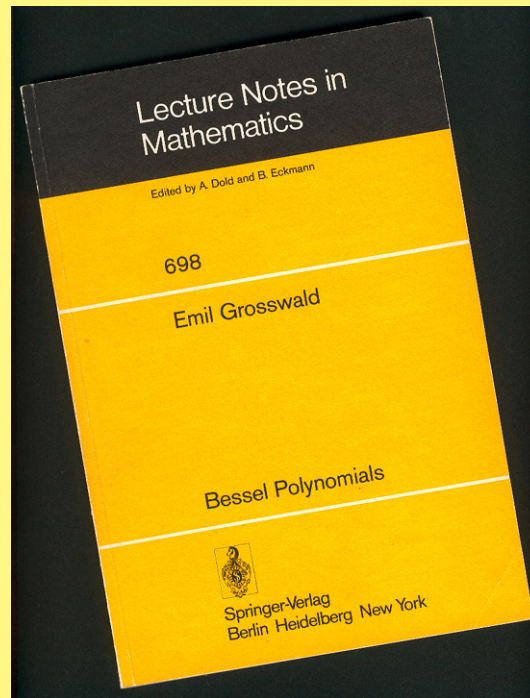
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Remark: In joint work with Ognian Trifonov this conjecture has now been resolved in the affirmative (to appear).

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  - Calvin and Hobbes

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- Legendre Polynomials (too hard)

The Laguerre Polynomials:

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$$L_n(x) = \frac{e^x}{n!} \frac{d^n \left(x^n e^{-x} \right)}{dx^n} = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n}{j} x^j$$

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Theorem 1 (I. Schur, 1929): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n \frac{x^n}{n!} + a_{n-1} \frac{x^{n-1}}{(n-1)!} + \cdots + a_1 x + a_0$$

is irreducible.

Theorem (1996): Let $a_0, a_1, \ldots, a_n$ denote arbitrary integers with $|a_0| = 1$, and let

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in which cases either $f(x)$ is irreducible or $f(x)$ is the product of two irreducible polynomials of equal degree. If $|a_n| = n$, then for some choice of $a_1, \ldots, a_{n-1} \in \mathbb{Z}$ and $a_0 = \pm 1$, we have that $f(x)$ is divisible by $x \pm 1$.

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$$L_n^{(0)}(x) = L_n(x) \qquad \text{(the Laguerre Polynomials)}$$

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Theorem 2 (I. Schur): Let n be a positive integer, and let $a_0, a_1, \cdots, a_n$ denote arbitrary integers with $|a_0| = |a_n| = 1$. Then

$$a_n \frac{x^n}{(n+1)!} + a_{n-1} \frac{x^{n-1}}{n!} + \cdots + a_1 \frac{x}{2} + a_0$$

is irreducible (over the rationals) unless $n = 2^r - 1 > 1$ (when $x \pm 2$ can be a factor) or $n = 8$ (when a quadratic factor is possible).

Theorem (joint with M. Allen): For n an integer ≥ 1, define
$$f(x) = \sum_{j=0}^{n} a_j \frac{x^j}{(j+1)!}$$
where the a_j's are arbitrary integers with $|a_0| = 1$. Write
$$n + 1 = k'2^u \quad \text{with } k' \text{ odd}$$
and
$$(n+1)n = k''2^v 3^w \quad \text{with } \gcd(k'', 6) = 1.$$
If
$$0 < |a_n| < \min\{k', k''\},$$
then $f(x)$ is irreducible.

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$$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6)$$

$$L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30)$$

$$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3 - 78x^2 + 1872x - 14040)$$

$$L_4^{(12/5)}(x) = \frac{1}{15000}(25x^2 - 420x + 1224)(25x^2 - 220x + 264)$$

$$L_5^{(39/5)}(x) = \frac{-1}{375000}(5x-84)\big(625x^4 - 29500x^3$$
$$+ \, 448400x^2 - 2662080x + 5233536\big)$$

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Theorem (joint with T.-Y. Lam): Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers n, the polynomial $L_n^{(\alpha)}(x)$ is irreducible over the rationals.

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A Special Case: $\alpha = n$

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- Hilbert's work and work of E. Noether (1918) began what has come to be known as Inverse Galois Theory.

- Van der Waerden showed that for "almost all" polynomials $f(x) \in \mathbb{Z}[x]$, the Galois group associated with $f(x)$ is the symmetric group S_n.

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- Schur did not find an explicit sequence of polynomials having Galois group A_n with $n \equiv 2 \pmod{4}$.

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Comment: Gow also showed that $L_n^{(n)}(x)$ is irreducible if

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Conjecture: If $n > 2$, then $L_n^{(n)}(x)$ is irreducible.

Theorem (joint work with R. Williams): For almost all positive integers n the polynomial $L_n^{(n)}(x)$ is irreducible (and, hence, has Galois group A_n for almost all $n \equiv 2 \pmod{4}$). More precisely, the number of $n \leq t$ such that $L_n^{(n)}(x)$ is reducible is

$$\ll \exp\left(\frac{9\log(2t)}{\log\log(2t)}\right).$$

Furthermore, for all but finitely many n, $L_n^{(n)}(x)$ is either irreducible or $L_n^{(n)}(x)$ is the product of a linear polynomial times an irreducible polynomial of degree $n-1$.

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Comment: The number of $n \leq t$ for which $f(x)$ is reducible for some choice of a_j as above is

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There exist explicit numbers α and $\beta > 0$ such that, for $n \geq \alpha$,

$$n(n+1) = 2^k 3^\ell m \implies m > n^\beta.$$

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The Ramanujan-Nagell equation
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Moreover, there exist explicit numbers α and $\beta > 0$ such that, for $x \geq \alpha$,

$$x^2 + 7 = 2^n m \implies m \geq x^\beta.$$

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$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n\left(e^{-x^2/2}\right)}{dx^n}$$

$$= \sum_{j=0}^{[n/2]} (-1)^n \binom{n}{2j} u_{2j}\, x^{m-2j}$$

where

$$u_{2j} = (2j-1)(2j-3)\cdots 3 \cdot 1$$

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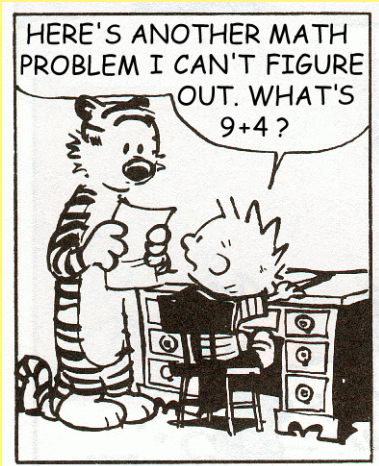
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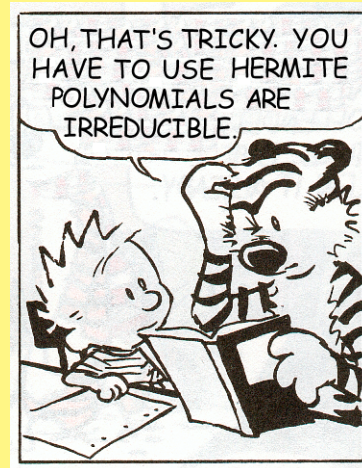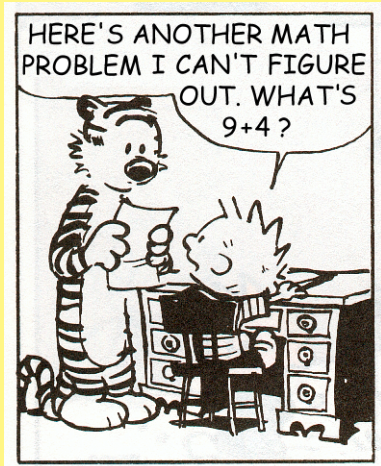
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is irreducible unless $2n$ is of the form $3^u - 1$ with $u > 1$.

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Email from Mark Kon:

Given a function $f(x)$, its wavelet transform consists of the family of functions $g(2^j x) * f(x)$, where g is the gaussian function, and j is an integer. The question was: if we know the zeroes of the second derivatives of this family of functions (over all j), can we recover f? ... The problem reduces to showing that none of these polynomials [certain Hermite polynomials] has zeroes (aside from the trivial one at the origin) which coincides with a zero of another one. So the bottom line is that the conjecture that f is uniquely recoverable follows from the non-overlapping of the zeroes of the Hermite polynomials.

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Then $p(x)$ is $(x-1)^3$ times an irreducible polynomial if n is even and $(x-1)^3(x+1)$ times an irreducible polynomial if n is odd.

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Joint Work With A. Borisov, T.-Y. Lam, O. Trifonov:

True for all but $O(t^{4/5+\varepsilon})$ values of $n \leq t$.

Theorem 3 (I. Schur, 1929): For $n > 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j x^{2j} / u_{2j}$$

is irreducible.

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Theorem (joint with M. Allen): For $n > 1$ and arbitrary integers a_j with $|a_0| = 1$ and

$$0 < |a_n| < 2n - 1,$$

the polynomial $f(x)$ above is irreducible for all but finitely many pairs (a_n, n).

Theorem 4 (I. Schur, 1929): For $n \geq 1$ and arbitrary integers a_j with $|a_0| = |a_n| = 1$, the polynomial

$$f(x) = \sum_{j=0}^{n} a_j x^{2j} / u_{2j+2}$$

is irreducible unless $2n$ is of the form $3^u - 1$ with $u > 1$.

Theorem (joint with M. Allen): For n an integer ≥ 1, define

$$f(x) = \sum_{j=0}^{n} a_j \frac{x^{2j}}{u_{2j+2}}$$

where the a_j's are arbitrary integers with $|a_0| = 1$. Write

$$2n + 1 = k'3^u \quad \text{with} \quad 3 \nmid k'$$

and

$$(2n + 1)(2n - 1) = k''3^v5^w \quad \text{with} \quad (k'', 15) = 1.$$

If

$$0 < |a_n| < \min\{k', k''\},$$

then $f(x)$ is irreducible for all but finitely many pairs (a_n, n).

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- O. Trifonov and M.F. have now shown that all Bessel polynomials are irreducible.

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Theorem (joint with O. Trifonov): If $a_0, a_1, \ldots, a_n$ are arbitrary integers with $|a_0| = |a_n| = 1$, then

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  A result of M.G. Dumas (in 1906) eliminates possible degrees for the factors of a polynomial using information about the divisibility of the coefficients by a given prime p (forming Newton polygons with respect to p).

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

    "Two such factorization schemes with a common, non-trivial factorization, will be called *compatible*. Otherwise, we call them incompatible. It is clear that if one can exhibit two incompatible factorization schemes, one thereby will have proved the irreducibility of the polynomial considered."

    Emil Grosswald
    Bessel Polynomials
    Lecture Notes Series

- Newton polygons are used to show that if certain conditions on divisibility by primes holds, then $f(x)$ is irreducible.

  Idea: To consider factorization schemes using many primes and show that they are incompatible. For a polynomial of degree n and a $k \in [1, n/2]$, find a prime p such that the Newton polygon with respect to p does not allow for a factor of $f(x)$ to have degree k.

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  For n large and k large (say $> n^{2/3}$), use that there are two primes in the interval $[n-k+1, n]$.

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  Now consider k small.

- Analysis to show that the conditions hold; usually this involves cases to eliminate possible factors depending on the size of their degrees.

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Problem n: 10, 12, and that's it.

- Analysis to show that the conditions hold; usually this involves cases to eliminate possible factors depending on the size of their degrees.

Lemma. For $3 \leq k \leq n/2$,

$$\prod_{\substack{p^r || n(n-1)\cdots(n-k+1) \\ p \geq k+1}} p^r > n$$

unless one of the following holds:

$$\begin{array}{lll} k = 3 & \text{and} & n = 6, 8, 9, 10, \text{ or } 18 \\ k = 4 & \text{and} & n = 9 \\ k = 5 & \text{and} & n = 10 \text{ or } 12. \end{array}$$

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J. Brillhart's Observation (1969):

$$6B_{11}(x) = x(x-1)(2x-1)(x^2-x-1)$$
$$\times (3x^6 - 9x^5 + 2x^4 + 11x^3 + 3x^2 - 10x - 5)$$

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Theorem (joint with A. Adelberg): A positive proportion of the polynomials $B_n^{(n)}(x)$ are Eisenstein (and, hence, irreducible).

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Theorem (joint with A. Adelberg): A positive proportion of the polynomials $B_n^{(n)}(x)$ are Eisenstein (and, hence, irreducible). More precisely, if the number of $n \leq t$ for which $B_n^{(n)}(x)$ is Eisenstein is $\mathcal{B}(t)$, then

$$\mathcal{B}(t) > t/5 \qquad \text{for } t \text{ sufficiently large.}$$

TIME FOR QUESTIONS