On the Factorization of $n(n+1)$

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Joint Work with M. Bennett & O. Trifonov

Part I: On the factorization of $x^2 + x$

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Problem: Can we narrow the gap between these ineffective and effective results?

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Theorem (R. Gow, 1989): If $n > 2$ is even and

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Work in Progress with Trifonov: We're attempting to show the irreducibility of $L_n^{(n)}(x)$ for all $n > 2$.

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$$x^2 + 7 = 2^n m$$

$$\left(\frac{x+\sqrt{-7}}{2}\right)\left(\frac{x-\sqrt{-7}}{2}\right) = \left(\frac{1+\sqrt{-7}}{2}\right)^{n-2}\left(\frac{1-\sqrt{-7}}{2}\right)^{n-2} m$$

Theorem: If x, n and m are positive integers satisfying

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$$m \geq \text{???}$$

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In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the 13^{th} power has absolute value ≈ 2.65 and the prime powers themselves have absolute value ≈ 90.51.