PROJECT SUMMARY
for Ognian Trifonov's Proposal

Several years ago, the proposer began his investigations into the use of finite differences to obtain results about gaps between squarefree numbers. This initial work led to him receiving the Distinguished Award of the Hardy-Ramanujan Society in 1991 (jointly with M. Filaseta). He also showed how the same approach could be applied to the analogous problem for gaps between k-free numbers. Subsequently, making further use of differencing techniques, the proposer in joint work with Filaseta and later in joint work with Huxley obtained improvements on known results for the distribution of squarefull numbers in short intervals. In a relatively recent paper, the proposer and Filaseta uniformized their approaches to these and other applications of differences and obtained two general theorems which estimate the number of lattice points close to a curve $y = f(x)$. They also showed how various applications (including a new one related to counting finite abelian groups) could be obtained from these two theorems. Similar work on the use of differences for counting lattice points close to a curve has also been done (and is currently being done) by Huxley and Sargos. The works of the proposer and these authors have motivated other applications of differences, including recent work by Konyagin on the least prime factor of a binomial coefficient and recent work by Pomerance and Canfield concerning a conjecture about Stirling numbers of the second kind. This proposal concerns advancing the use of differences to obtain further improvements on estimates for the number of lattice points close to a curve, to apply such improvements to the known applications of differences, to generalize some of the existing approaches (for example, to count lattice points close to a surface $z = f(x, y)$), and to obtain new applications as a result of this work (for example, information on the distribution of cubefull numbers in short intervals).

PROJECT DESCRIPTION

Estimating the Number of Lattice Points Close to a Curve

1. Background to the Proposer's Recent Work Using Finite Differences

Recently, in [12], the proposer and Filaseta made use of finite differences to obtain upper bounds for the size of the set $S = \{n \in (N, 2N] \cap \mathbb{Z} : \|f(n)\| < \delta\}$, where f is a function satisfying certain conditions on its derivatives, $\delta > 0$, and $\|x\| = \min\{|x - m| : m \in \mathbb{Z}\}$. When δ is small, the finite difference approach produces upper bounds that improve on estimates obtainable by current exponential sum techniques. Various applications are given in [12] related to gaps in number theoretic sequences. Indeed, this paper developed largely as a consequence of the proposer's joint work with Filaseta on gaps between squarefree numbers in [10] (which led to them jointly receiving the 1991 Distinguished Award of the Hardy-Ramanujan Society), the proposer's work on gaps between k-free numbers in [55], and the proposer's joint work with Filaseta on the distribution of squarefull numbers in short intervals in [11]. Further difference approaches for estimating $|S|$ have also been developed by Huxley, Konyagin, Sargos, and the proposer (cf. [24,26,27,28,32,56]).

In addition to the applications mentioned above, the use of differences has proven to be valuable for other number theoretic problems. Differencing techniques have been used (see [12]) to obtain improvements on a problem of Erdős [3] concerning moments of gaps between squarefree numbers; the best result to date is due to Huxley [25] making use of a combination of new clever ideas for the problem and the use of difference estimates such as those contained in [12]. Konyagin [32] has recently added a new idea into the difference methods and applied his estimates for $|S|$ to obtain an improvement on work of Granville and Ramaré [19] associated with the smallest prime factor of $\binom{n}{k}$. Huxley and the proposer [28] have extended work of Swinnerton-Dyer [51] to obtain a new difference approach with applications to the distribution of squarefull numbers. Filaseta has shown how a method of Nair [39,40] can be viewed as a difference approach in algebraic number fields and used it to obtain independent results about k-free values of irreducible polynomials [7] and binary forms [8]. In [12], the proposer and Filaseta also show how difference methods can be applied to a gap problem associated with a counting function for finite abelian groups. And finally, Pomerance (private communication) has indicated that he is doing joint work with Canfield making use of difference estimates to show that a certain conjecture about Stirling numbers of the second kind holds for "most" positive integers.

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As an example of the general type of estimate that has been obtained by the use of differences, we mention the following result from [12].

Theorem 1. *Let $N > 1$. Let r be an integer ≥ 3. Let T be a positive real number. Suppose that f is a function with at least r derivatives and with $f^{(j)}(u) \asymp TN^{-j}$ for $j \in \{r-2, r-1, r\}$ and $u \in (N, 2N]$. Let δ be a positive real number with*

$$\delta < k \min\{TN^{-r+2}, T^{(r-4)/(r-2)}N^{-r+3} + TN^{-r+1}\},$$

for some sufficiently small constant k depending on r and the implied constants in the asymptotic formulas $f^{(j)}(u) \asymp TN^{-j}$ above. Let

$$S = \{u \in (N, 2N] \cap \mathbb{Z} : ||f(u)|| < \delta\}.$$

Then

$$|S| \ll T^{2/(r(r+1))}N^{(r-1)/(r+1)} + N\delta^{2/((r-1)(r-2))} + N\left(\delta TN^{1-r}\right)^{1/(r^2-3r+4)}.$$

The main goal of this proposal is to investigate areas in which results, such as this, can be improved or generalized and to consider certain applications that might be produced from improvements on known estimates of $|S|$.

2. The Proposed Projects

In this section, we outline various aspects of the recent research on the use of differences to make estimates of the type given in Theorem 1 and of their applications that the proposer plans to pursue during the next few years.

Recently, Huxley and the proposer [28] obtained a new estimate for $|S|$ based on generalizing a technique of Swinnerton-Dyer [51]. The idea behind the new estimate is to first reduce the problem of counting points P close to a curve $y = f(x)$ to that of counting quadruples of points P_1, P_2, P_3, P_4 close to a curve with the triples P_1, P_2, P_3 and P_2, P_3, P_4 being noncollinear. Suppose the four points are (m_j, n_j), for $1 \leq j \leq 4$. One bounds the number of possible choices for the eight variables m_1, m_2, m_3, m_4 and n_1, n_2, n_3, n_4 by considering the new quantities

$$a_1 = m_2 - m_1, \quad a_2 = m_3 - m_2, \quad a_3 = m_4 - m_3,$$

$$b_1 = n_2 - n_1, \quad b_2 = n_3 - n_2, \quad b_3 = n_4 - n_3,$$

together with

$$D_1 = a_2b_3 - a_3b_2, \quad D_2 = a_1b_3 - a_3b_1, \quad D_3 = a_1b_2 - a_2b_1.$$

The three expressions D_1, D_2, D_3 represent divided differences and, hence, are somewhat small (certainly much smaller than $\max\{a_j\}^2$ provided that $|f''|$ is small). One can find a good estimate for the number of values of the triple (D_1, D_2, D_3). For each fixed triple, one then bounds the number of possible values for the sextuple $(a_1, a_2, a_3, b_1, b_2, b_3)$. This is simplified somewhat by the relations

$$a_1D_1 - a_2D_2 + a_3D_3 = 0 \quad \text{and} \quad b_1D_1 - b_2D_2 + b_3D_3 = 0$$

which imply that for a fixed triple (D_1, D_2, D_3) and say a_1, a_2, b_1, b_2, the values of a_3 and b_3 are determined. The idea is then that bounding the number of choices for the eight variables m_1, m_2, m_3, m_4 and n_1, n_2, n_3, n_4 has been replaced with bounding the number of choices for seven variables, namely $D_1, D_2, D_3, a_1, a_2, b_1, b_2$. Such estimates are made in [28] to derive at a new bound for $|S|$ as in Theorem 1.

A current drawback of this approach is that it makes use of divided difference approximations for $f''(x)$ (to estimate the size of the D_j) and bounds on $f'''(x)$ (in estimating the number of choices for the seven variables above), but there is currently no known generalization of the final estimates on $|S|$ based on similar techniques but with use of higher derivatives. This is uncommon of other difference approaches as well as exponential sum approaches, so it is reasonable to expect that some such generalization of these methods does exist.

Problem 1. Generalize the approach of Huxley and the proposer in [28] to take advantage of knowledge of higher derivatives of $f(x)$.

During the last year, the proposer has begun making some progress toward resolving the above problem. He has shown $|S| \ll T^{5/33} N^{17/33}$ when $\delta \ll T^{1/3} N^{-4/3}$. This estimate improves on the known estimates when $T = N^{\alpha}$ and α is close to $3/2$. The proof is based on an extention of the method of Swinnerton-Dyer making use of the size of $f'''(x)$ and of the existence of $f^{(4)}(x)$. There are several ways to refine this method further and to relax the condition on δ. The proposer has obtained some new estimates for $|S|$ when $T = N^{\alpha}$ when α is close to $k + 1/2$ where k is a positive integer. He plans to work toward further improvements in this direction.

One of the applications of this technique is to that of determining $h = h(x)$ as small as possible such that the density of squarefull numbers in $(x, x+h]$ is asymptotically $\zeta(3/2)h/(2\zeta(3)\sqrt{x})$. Combining the new estimate above from [28] with Theorem 1, Huxley and the proposer show that the correct asymptotics can be obtained for $h = x^{(5/8)+\varepsilon}$ with $\varepsilon > 0$ arbitrary. The basic idea is that one wants to count pairs (a, b) for which $x < a^2 b^3 \le x + h$. This inequality is rewritten as

$$\sqrt{\frac{x}{b^3}} < a \le \sqrt{\frac{x+h}{b^3}} < \sqrt{\frac{x}{b^3}} + O\left(\frac{h}{\sqrt{xb^3}}\right).$$

In terms of Theorem 1, one considers $f(u) = \sqrt{x/u^3}$ and estimates the number of $b \in (N, 2N]$ by considering $u \in (N, 2N]$ for which $\|f(u)\| < \delta$ where $\delta = ch/\sqrt{xN^3}$. Theorem 1 is used in [28] to make such an estimate when $N \le x^{1/6}$. Here, one gets $|S| \ll x^{1/8}$. A similar estimate is obtained for larger values of N by making use of the new bounds on $|S|$ in [28]. In both cases, however, the estimates can be sharpened if N is not too close to $x^{1/6}$. In other words, if $\varepsilon > 0$ and $N \ll x^{(1/6)-\varepsilon}$, then Theorem 1 gives a better bound on $|S|$. If $\varepsilon > 0$ and $N \gg x^{(1/6)+\varepsilon}$, then the new result in [28] gives a better bound on $|S|$. A generalization to the main estimate in [28] as described in the problem above will likely improve on the estimate for $|S|$ in this problem when N is near $x^{1/6}$ (this comment can be justified more rigorously, but we do not bother with the details here). Thus, in dealing with the problem above, the proposer will also keep in mind the following.

Problem 2. Obtain improvements on results about the distribution of squarefull numbers in short intervals (taking advantage of any estimates obtained in the previous problem).

As mentioned above, the work of Huxley and the proposer in [28] arose in generalizing a method of Swinnerton-Dyer [51]. To be more precise, Swinnerton-Dyer's paper is centered on obtaining an estimate for the number of points on a curve rather than close to a curve. Therefore, in his paper, he obtains an estimate of the type given in Theorem 1 with, however, $\delta = 0$. Huxley and the proposer generalize the approach to deal with $\delta > 0$. Following the basic idea of Swinnerton-Dyer, they consider triples of points P_1, P_2, P_3 close to a curve which are non-collinear (this prevents the D_j above from being 0). One can generalize this idea and consider instead $n+2$ points which do not lie on an algebraic curve of degree n (rather than simply being non-collinear). This has in fact been accomplished rather definitively by Bombieri and Pila [2] in the case that $\delta = 0$. What has not yet been accomplished, however, is a general estimate when $\delta > 0$ for algebraic curves of degree > 1. The proposer plans to investigate the possibility of modifying the approach of Bombieri and Pila to obtain new estimates for $|S|$ (when $\delta > 0$). Even in the case of using algebraic curves of degree 2 such an estimate would be of interest.

Problem 3. Obtain new estimates for $|S|$ using information obtained by considering $n+2$ points close to a curve $y = f(x)$ that do not lie on an algebraic curve of degree n; in particular, analyze the situation when n is small but > 1.

The proposer anticipates that such improvements will play a greater role in applications currently requiring knowledge about large derivatives of $f(x)$. Such applications currently include estimating the size of the gaps between k-free numbers, obtaining results about the moments of such gaps, and estimating the size of the smallest prime factor of $\binom{n}{k}$.

Some progress in the direction of Problem 3 has been made in the case of dealing with quadratic curves of the form $y = ax^2 + bx + c$ when $\delta > 0$. This was done by Huxley [26], resulting in an improvement on his joint work with the proposer [28], and independently by the proposer [56], leading to an improvement to Theorem 1 in the case $r = 3$ and $r = 4$. In both cases, the (seemingly) main terms for the previous estimates remain untouched but the conditions on δ are allowed to be relaxed considerably. The particular approach in [56] made use of the following: if $c_1TN^{-3} \le f'''(u) \le c_2TN^{-3}$ for all $u \in (N, 2N]$ for some positive constants c_1 and c_2 and if $\delta < kT^{1/3}N^{-4/3}$ where k is a sufficiently small constant depending on c_1 and c_2, then

$$|S| \ll |S_1| + \delta N^{\frac{1}{3}} + N^{\frac{1}{2}},$$

where S_1 is a strictly 3-convex subset of S and the distance between any two distinct elements of S_1 is $\gg N^{1/2}T^{-1/6}$. Thus, in many cases we can work with a subset of S that has the same size as S but is 3-convex and well-spaced. This is also a prerequisite for extending Swinnerton-Dyer's work [51]. More results concerning the geometry of S (n-convexity and spacing) can be obtained by working with higher derivatives of f. The proposer plans to improve on previous work by using such information. In particular, he would like to extend the approach in [56] to improve on Theorem 1. At present the existing algorithms for obtaining a n-convex set from a finite set by discarding some of its elements are somewhat crude. The proposer plans to work on obtaining better algorithms based on non-linear selection of the elements that have to be discarded. An eventual success will lead not only to new results in number theory, but to applications in approximation theory and in certain classes of partial differential equations as well.

Problem 4. Relax the condition on δ in Theorem 1 or eliminate it completely if possible.

Divided differences have proven to be an important tool for obtaining certain estimates for $|S|$ (such as Theorem 1). One way to view divided differences is the following. Let $u_0, u_1, \ldots, u_r$ be distinct points in $(N, 2N]$. Define the r-th divided difference $f[u_0, u_1, \ldots, u_r]$ as $a_0 f(u_0)+a_1 f(u_1)+\ldots+a_r f(u_r)$ where $(a_0, a_1, \ldots, a_r)$ is the unique solution to the system of linear equations

$$\sum_{i=0}^{r} a_i u_i^k = 0 \text{ for } k \in \{0, 1, \ldots, r-1\} \quad \text{and} \quad \sum_{i=0}^{r} a_i u_i^r = 1.$$

These equations imply that if f has a continuous r-th derivative then $a_0 f(u_0) + a_1 f(u_1) + \ldots + a_r f(u_r) = f^{(r)}(\xi)/r!$ where ξ is in the the interval spanned by the points $u_0, u_1, \ldots, u_r$. A bad feature of divided differences is that in many cases when the numbers u_i are all integers then the coefficients a_i are rational numbers with numerators and denominators that are rather large. Konyagin [32] obtained a new estimate for $|S|$ by considering more numbers u_i to get a linear combination of values of f equal to a value of $f^{(r)}$ with the coefficients a_i having relatively small numerators and denominators. One can formulate the problem Konyagin deals with as follows: Let $s > r$. Consider the following system

$$\sum_{i=0}^{s} a_i u_i^k = 0 \text{ for } k \in \{0, 1, \ldots, r-1\} \quad \text{and} \quad \sum_{i=0}^{s} a_i u_i^r \neq 0.$$

Find a solution $(a_0, a_1, \ldots, a_r)$ such that all a_j's are integers and $\mathcal{M} = \sum_{i=0}^{s} {a_i}^2$ is minimal. It is clear that the lattice points satisfying the first r equations above form an $s-r$ dimensional lattice. Using properties of lattices, Konyagin was able to obtain good estimates for $\mathcal{M}$ when $s = 2r$ and r is large. This allows him to improve the existing estimates for $|S|$ when $f^{(r)}(u) \asymp 1$ for $u \in (N, 2N]$ and r is large. His approach does not work well when $N \le T \le N^2$. By using known facts about the geometry of the set S one can improve Konyagin's estimates in the above range. The case $N \le T \le N^2$ is critical for most applications of estimates for $|S|$, like distribution of squarefree and squarefull numbers in short intervals and moments of gaps between k-free numbers. It appears that modifying Konyagin's approach is a promising way to improve known results for such applications.

Problem 5. Modify Konyagin's approach in [32] to improve the existing estimates for $|S|$ when $N \le T \le N^2$.

To clarify, which estimate (whether obtained by a finite difference approach or by exponential sums) for the size of $|S|$ works best on a particular problem depends on the relative sizes of T and N as well as δ. Our goal in Problem 5 is to find improvements over existing results for certain values of T and N as above.

In [14] Fouvry and Iwaniec developed a method for estimating exponential sums with monomials. Their method uses, in a crucial way estimates for certain spacing problems. They can be regarded as estimates for the number of integer points close to a surface. Recently, Sargos and Wu [44] improved on [14] by obtaining better estimates for these spacing problems. As an application they obtained new results on the distribution of almost prime numbers in short intervals and on the distribution of B-free numbers in short intervals.

The proposer also plans to study the problem of using differences to estimate the number of lattice points close to a surface, i.e., the size of the set

$$S' = \{(u,v) \in \mathbb{Z}^2 : u \in (N,2N], v \in (M,2M], ||f(u,v)|| < \delta\}$$

for some appropriately behaved function $f(u,v)$. In other words, he will consider the possibility of extending the difference techniques to obtain multi-dimensional versions of Theorem 1.

Problem 6. Find a good upper bound (using differences) for the size of the set S' mentioned above. Obtain new bounds for exponential sums with monomials and new results on the distibution of almost prime numbers and B-free numbers. More generally, obtain a version of Theorem 1 for functions of several variables.

Let $C(x)$ denote the number of cubefull numbers $\leq x$. An application of an upper bound for $|S'|$ would be to determining $h = h(x)$ as small as possible for which $C(x+h) - C(x)$ has the appropriate asymptotic size. Our initial work on this problem has led to a simple proof that one can take $h = x^{(2/3)+\theta}$ with $\theta = 1/8$ and a more complicated argument that one can do better. Our results are not yet satisfactory as Liu [38] has obtained the better result $\theta = 11/92$ using exponential sum techniques. The idea would be to see what differencing methods produce for such a result, but it would clearly be reasonable to also consider exponential sum approaches. Perhaps the best approach in the end might be to consider a combination of the two techniques. An approach that makes use of differences here would be of interest partially because the application of differences to cube-full numbers has not yet been realized (in a form that improves on other known techniques).

Problem 7. Using results from the previous problem, determine a small value of θ for which

$$C(x + x^{2/3+\theta}) - C(x) = Ax^{2/3+\theta}\,(1 + o(1))\,,$$

where A is an appropriate constant.

Nair [39] showed how the ideas of Halberstam and Roth [20] could be used in algebraic number fields to obtain information about k-free values of polynomials. Further work in this direction was done by Nair in [40] and by Huxley and Nair [23]. That the approach of Halberstam and Roth could be viewed as a modified difference approach was first elaborated on by Filaseta [4] where he extended the approach to further differences and improved on knowledge about gaps between k-free numbers. Later the approach was modified further to the already mentioned results about gaps between squarefree numbers by the proposer and Filaseta [10] and the corresponding result for k-free numbers by the proposer [55]. More recently, the approach of Nair was used by Filaseta [8] to obtain information about k-free values of binary forms. A more elaborate use of differences in algebraic number fields was shown by Filaseta [7] to imply improvements on known results about gaps between m for which $f(m)$ is k-free where $f(x)$ denotes an irreducible polynomial. The proposer is interested in taking advantage of the new difference approaches that have been developing as discussed earlier in this proposal and extending the knowledge of what can be done in algebraic number fields. In particular, is there a generalization of the Swinnerton-Dyer approach to algebraic number fields and is it possible to prove that for $f(x) = x^4 + 1$ there are infinitely many m for which $f(m)$ is squarefree?

Problem 8. Extend the current difference approaches to algebraic number fields.

Thus far the techniques described above have been used to give information about the number of $n \in (N, 2N]$ with $\{f(n)\} \in I$ for some small interval $I \subseteq [0,1)$. In many cases, one would like more.

Problem 9. Obtain new information about the general distribution of $\{f(n)\}$ in $[0,1)$.

In particular, the proposer will try to combine the various differencing ideas used to obtain the results above with an elementary approach of Vinogradov (cf. [15]) for the Dirichlet divisor problem. Vinogradov showed that

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x),$$

where γ is Euler's constant and $\Delta(x) \ll x^{1/3}$. Although better estimates for $\Delta(x)$ exist, the proposer will try to use finite difference methods together with the techniques of Vinogradov to obtain elementary improvements on the estimate for $\Delta(x)$. In particular, he would like to see if an elementary proof can be given to show that $\Delta(x) \ll x^{(1/3)-\epsilon}$ for some $\epsilon > 0$.

The proposer is also interested in incorporating the use of finite differences directly into exponential sum techniques. One apparently needs to arrange for such differences to occur in the argument of $e(x) = e^{2\pi i x}$. Typically, one encounters finite differences when obtaining exponential sum estimates (cf. [17]), but to take advantage of the particular differencing techniques used to obtain results like Theorem 1, the differences need to be of a different type. For example, one would like to be able to obtain divided differences in the argument of $e(x)$. Such differences have not as yet been used to make exponential sum estimates (and it is not completely clear how they can be used), so the proposer will explore possible methods for incorporating such differences to estimate exponential sums.

Problem 10. Incorporate the finite difference approaches into the theory of exponential sums and develop new exponential sum estimates (for example, derive new exponent pairs).

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