Classical use of exponential sums have been applied to give a variety of results about the distribution of multiplicative functions in short intervals. Among the contributions to the subject are the works of S. W. Graham and G. Kolesnik [11], D. R. Heath-Brown [12], H. Li [24], A. Ivić [18], C.-H. Jia [19,20], E. Krätzel [23], H. Liu [25,26], R. A. Rankin [29], H. E. Richert [30], K. F. Roth [31], P. G. Schmidt [32,33,34], and P. Shiu [35,36]. These results have seen recent advances through the use of finite difference techniques developed by M. N. Huxley [13-17], S. Konyagin [21], Sargos [16], and both independently and jointly by the two PI's Filaseta and Trifonov (cf. [4-10,40-44]). For example, they have investigated estimates for the sums

$$\sum_{x<n\le x+h} F(n) \tag{1}$$

where h is small ($h = x^\theta$ for some $\theta \in (0,1)$ with the goal of minimizing θ) and $F(n)$ represents one of the three characteristic functions for the set of k-free numbers, for the set of squarefull numbers, and for the set of n for which the number of non-isomorphic abelian groups of order n is a given fixed number k. In fact, each of the papers indicated as applications of exponential sums listed above (from the paper by Graham and Kolesnik [11] to the papers by Shiu [35,36]) give estimates for (1) for one of these three characteristic functions. In [10], Filaseta and Trifonov show how differences can be used in each of these three cases to establish improvements over these known exponential sum estimates.

One problem the PI's would pursue is to generalize the applications of finite differences to more general functions $F(n)$. More precisely, we pose to investigate the following.

Problem 1. Using the recently developed differencing techniques, obtain an estimate for (1) for general $F(n)$ (subject to as few constraints as possible).

Such $F(n)$ need not necessarily be multiplicative functions, but clearly such a constraint might lead to a better estimate or be advantageous to the approaches. That a general result of some sort should exist is already suggested by the treatment of the non-isomorphic abelian group problem by Filaseta and Trifonov in [10]. Further recent evidence includes the observation by Anguel Kumchev [22], a current student of Filaseta, that differences can be used to obtain a short interval result when $F(n)$ denotes the number of squarefree divisors of n. Also, Trifonov (unpublished) observed that one can use differences to obtain a short interval result when $F(n)$ is the number of "exponential" divisors of n (see the work of M. V. Subbarao [38] and J. Wu [45] for asymptotics concerning exponential divisors). Trifonov's approach here seems to naturally lend itself to possible generalizations. A general result might also address the function $F(n)$ defined by

$$F(p^j) = \begin{cases} 0 & \text{if } j < r \\ 1 & \text{if } r \le j \le s \\ 0 & \text{if } j > s \end{cases}$$

where r and s are integers satisfying $s > r \geq 1$. This is suggested by the characteristic function for the squarefree numbers (the case $r = 1$ and $s = 1$) and the characteristic function for the squarefull numbers (the case $r = 2$ and $s = \infty$). This function is discussed, for example, in the work of H. Menzer [28]. We mention such examples of $F(n)$ reluctantly. The point of this investigation is not to obtain results for various special classes of functions $F(n)$; the focus is rather to obtain a result that includes as general a class of multiplicative functions as possible.

Problems involving short interval results of the type in (1) are closely related to estimating the size of the set

$$S = \{n \in (N, 2N] \cap \mathbb{Z} : \|f(n)\| \leq \delta\}, \tag{2}$$

where f is a function satisfying certain conditions on its derivatives, $\delta > 0$, and $\|x\| = \min\{|x - m| : m \in \mathbb{Z}\}$. When δ is small, the finite difference approaches, as in the applications discussed above, produce upper bounds that improve on estimates obtainable by current exponential sum techniques. Different differencing techniques apply to different problems depending on the function f and the size of N. As an example, we consider the case that $F(n)$ is the characteristic function for the squarefree numbers. One estimates (1) by showing that typically $F(n) = 1$ unless n is divisible by the square of a small prime. This is obvious if one considers a full range of n up to some number x, but treating the case of n in a short interval $(x, x+h]$ is a more difficult problem. This short interval problem for squarefree numbers boils down to establishing that there are $o(h)$ primes $p > h$ such that $p^2|n$ for some $n \in (x, x + h]$. For each $p > h$ such that $p^2|n$, we have $n = p^2 m \in (x, x + h]$ for some integer m and, hence,

$$\frac{x}{p^2} < m \leq \frac{x}{p^2} + \frac{h}{p^2}.$$

Thus, if we consider $f(u) = x/u^2$ and $\delta = h/N^2$, then the size of the set S in (2) gives an upper bound for the number of primes $p \in (N, 2N]$ satisfying $p^2|n$ for some $n \in (x, x + h]$. As N varies, the short interval problem for squarefree numbers is reduced to an estimate for $|S|$.

In the case of the characteristic function for squarefull numbers, improvements over the work in [10] by further differencing methods have been made by Huxley and Trifonov [17] and by Konyagin and Trifonov (in progress). The first of these showed how a divided difference approach of H. P. F. Swinnerton-Dyer [39] for estimating the number of points on a curve could be extended to an estimate for the number of lattice points close to a curve. In other words, Swinnerton-Dyer obtained estimates in the case $\delta = 0$ in (2) above, and Huxley and Trifonov extended his approach to $\delta > 0$ to give an improvement in the short interval result for squarefull numbers. Swinnerton-Dyer's method relies on the use of a third order divided difference and certain convexity conditions. Extending the approach to divided differences of higher order required a bit of work largely because of issues associated with handling the convexity condition. Making use of some ideas of Huxley [15], Konyagin and Trifonov have bypassed the issues of convexity and extended the work of Swinnerton-Dyer to fourth order divided differences.

Problem 2. Generalize the approach of Konyagin and Trifonov to divided differences of an arbitrary order and apply the new estimates to short interval results for multiplicative functions.

An arithmetic problem that arises in the Konyagin-Trifonov approach is to estimate the number of 8-tuples $(D_1, D_2, D_3, D_4, z_1, z_2, z_3, z_4)$ where the D_j are integers satisfying $0 < |D_j| \leq B$ and $D_1 + D_3 \neq D_2 + D_4$ and the z_j are integers with $0 < z_1 < z_2 < z_3 < z_4 \leq A$ satisfying

$$D_1 z_1 - D_2 z_2 + D_3 z_3 - D_4 z_4 = 0$$

and

$$D_1 z_1^2 - D_2 z_2^2 + D_3 z_3^2 - D_4 z_4^2 = 0.$$

With a little work one can obtain the bound $O(A^{1+\epsilon} B^4)$, but one expects something closer to $O(AB^2)$. We know how to decrease the exponent on B from 4 to 11/3, but more work is needed on this problem as any improvement on this estimate leads to a sharper estimate for $|S|$ in (2). Such an estimate would in turn lead to an improvement in the application of differences to the squarefull problem for short intervals. In fact, these results also give improvements to Swinnerton-Dyer's original work where $\delta = 0$.

Problem 3. Make further progress on estimates for the number of 8-tuples

$$(D_1, D_2, D_3, D_4, z_1, z_2, z_3, z_4)$$

as described above.

In dealing with Problem 2, we would also investigate further problems of the type just described. More precisely, further order divided differences leads to the problem of estimating the number of $2n$-tuples of integers $D_1, \ldots, D_n$ and $z_1, \ldots, z_n$ with $0 < |D_j| \leq B$ for each j, $0 < z_1 < z_2 < \cdots < z_n \leq A$, $\sum_{k=1}^{n} (-1)^k D_k \neq 0$, and $\sum_{k=1}^{n} (-1)^k D_k z_k^j = 0$ for $j = 1, 2, \ldots, n-2$. In each of these cases, we are mainly interested in an upper bound when B is "small" compared to A.

Suppose $F(n) = 1$ if and only if n is the sum of two squares. Again, F is multiplicative. It would be of particular interest in this case to find a *lower* bound for the sum in (1). A lower bound result would correspond to a short interval result for numbers which are the sum of two squares. In this case, a very simple argument shows that between x and $x + O(x^{1/4})$ there is a number which is the sum of two squares and, although this result dates back to R. P. Bambah and S. Chowla [1] in 1947, no improvement on the exponent $1/4$ has ever been obtained.

Problem 4. Obtain lower bound estimates for the sum given in (1).

A related question concerning gaps between lattice points on circles and ellipses has been more recently considered by J. Cilleruelo and A. Córdoba [2, 3]. This problem would correspond to upper bound estimates for sums of the type given in (1), and the PI's plan to consider possible applications of the recent differencing approaches to this work of Cilleruelo and Córdoba.

The PI's also plan to study the problem of using differences to estimate the size of the set

$$S' = \{(u, v) \in \mathbb{Z}^2 : u \in (N, 2N], v \in (M, 2M], ||f(u, v)|| < \delta\}$$

for some appropriately behaved function $f(u, v)$. In other words, we will consider the possibility of extending the current differencing techniques to obtain multi-dimensional versions of the type already investigated.

Problem 5. Find a good upper bound (using differences) for the size of the set S' mentioned above or, more generally, obtain estimates for the number of lattice close to a surface in higher dimensions.

It is anticipated that such estimates would then be applied to obtain new short interval results for multiplicative functions. For example, if $F(n)$ is the characteristic function for cubefull numbers, then estimates for $|S'|$ would lead to estimates for the size of the sum in (1). In other words, if $C(x)$ denotes the number of cubefull numbers $\leq x$, estimates for $|S'|$ lead to estimates for $C(x + h) - C(x)$. Our initial work on this problem has led to a simple proof that one can take $h = x^{(2/3)+\theta}$ with $\theta = 1/8$ and a more complicated argument that one can do better. Our results are not yet satisfactory as Liu [27] has obtained the better result $\theta = 11/92$ using exponential sum techniques. The idea would be to see what differencing methods produce for such a result, but it would clearly be reasonable to also consider exponential sum approaches. Perhaps the best approach in the end might be to consider a combination of the two techniques. An approach that makes use of differences here would be of interest partially because the application of differences to cubefull numbers has not yet been realized (at least in a form that improves on other known techniques).

Problem 6. Using results from the previous problem, determine a small value of θ for which

$$C(x + x^{2/3+\theta}) - C(x) = Ax^{2/3+\theta} (1 + o(1)),$$

where A is an appropriate constant.

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