ON GAPS BETWEEN SQUAREFREE NUMBERS II

Michael Filaseta*and Ognian Trifonov**

Dedicated to the memory of Vasil Popov

1. Introduction.

In this paper, the authors continue their work on the problem of finding an $h = h(x)$ as small as possible such that for x sufficiently large, there is a squarefree number in the interval $(x, x+h]$. This problem has been investigated by Fogels [4], Roth [11], Richert [10], Rankin [9], Schmidt [12], Graham and Kolesnik [5], the second author [14,15], and the first author [2]. In particular, the authors [3] have recently shown by elementary means that there is a constant $c > 0$ such that for x sufficiently large, the interval $(x, x+h]$ with $h = cx^{8/37}$ contains a squarefree number. Using exponential sums, they showed that $8/37$ may be replaced by $3/14$. A more extensive history of the problem can be found in their paper [3]. The purpose of this paper is to make the following improvement.

Theorem. *There exists a constant $c > 0$ such that for x sufficiently large the interval $(x, x + cx^{1/5} \log x]$ contains a squarefree number.*

The proof of the Theorem will be elementary. Much of the ground work has already been done and described in previous work on the problem (cf. [3]). For the purposes of completeness, we will present most of the necessary background here while at the same time introducing new approaches to some of the previous work.

2. Preliminaries

Notation (unless specified otherwise):

c is a sufficiently large constant. For convenience, we consider $c \geq 1$.
x is a sufficiently large real number (i.e., $x \geq x_0$ for some $x_0 = x_0(c)$).

---

*Research was supported in part by the NSF under grant number DMS-8903123.

**This research was done while the author was visiting the University of South Florida and was supported in part by Contract 50 with the Committee of Science, Bulgaria.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

a, b, d, and u denote positive integers.
p denotes a prime.
$\theta = 1/5$.
$h = cx^\theta \log x$.
$f(x) \ll g(x)$ means that there is a constant c' independent of c such that $|f(x)| \le c' g(x)$ for all x sufficiently large.
ϕ is a number $> \theta$. More specifically, $x^\phi > h\sqrt{\log x}$.
α is a real number.
v_1, v_2, and β are positive real numbers.

Let S denote the number of integers in $(x, x+h]$ which are not squarefree. For d a positive integer, let M_d denote the number of multiples of d^2 in the interval $(x, x+h]$. In particular, since x is sufficiently large and $h \le x^{1/4}$, we get that if $d > 2\sqrt{x}$, then $M_d = 0$. We easily get that

$$S \le \sum_{p \le 2\sqrt{x}} M_p.$$

Hence,

$$S \le S_1 + S_2,$$

where

$$S_1 = \sum_{p \le h\sqrt{\log x}} M_p \quad \text{and} \quad S_2 = \sum_{h\sqrt{\log x} < p \le 2\sqrt{x}} M_p.$$

Since $M_p \le (h/p^2) + 1$, we get that

$$\begin{aligned} S_1 &\le \sum_{p \le h\sqrt{\log x}} \left(\frac{h}{p^2} + 1 \right) \\ &< h \sum_{n=2}^{\infty} \frac{1}{n^2} + \pi(h\sqrt{\log x}) \\ &= h(\frac{\pi^2}{6} - 1) + \pi(h\sqrt{\log x}). \end{aligned}$$

By the prime number theorem or a Chebyshev estimate, we get that

$$S_1 \le \frac{2}{3} h.$$

Therefore, in order to prove the Theorem, it suffices to show that $S_2 \ll c^\sigma x^\theta \log x$ for some $\sigma < 1$.

Define

$$S(t_1, t_2) = \{d \in (t_1, t_2] : \exists \text{ an integer } m \text{ such that } md^2 \in (x, x+h]\}.$$

Suppose that $d \in (h\sqrt{\log x}, 2\sqrt{x}]$. Then $d^2 > h$ so that there is at most one multiple of d^2 in $(x, x+h]$. We get that for $d \in (h\sqrt{\log x}, 2\sqrt{x}]$

$$M_d = 1 \iff d \in S(h\sqrt{\log x}, 2\sqrt{x})$$

and

$$M_d = 0 \iff d \notin S(h\sqrt{\log x}, 2\sqrt{x}).$$

Therefore, $S_2 \leq |S(h\sqrt{\log x}, 2\sqrt{x})|$. To get bounds for $|S(h\sqrt{\log x}, 2\sqrt{x})|$ we will use the following lemma.

Lemma 1. *If*

$$|S(x^\phi, 2x^\phi)| \ll x^{\alpha-\beta\phi} \quad \textit{for } v_1 \leq \phi \leq v_2,$$

then

$$|S(x^{v_1}, 2x^{v_2})| \ll_\beta x^{\alpha-\beta v_1}.$$

If

$$|S(x^\phi, 2x^\phi)| \ll x^{\alpha+\beta\phi} \quad \textit{for } v_1 \leq \phi \leq v_2,$$

then

$$|S(x^{v_1}, 2x^{v_2})| \ll_\beta x^{\alpha+\beta v_2}.$$

A proof of the above lemma is fairly simple and can be found in [1]. The lemma is essentially contained in Roth's paper [11]. We note that if one replaces β with 0 in the first asymptotic inequality in Lemma 1, then the second asymptotic inequality holds with β replaced by 0 and with an extra factor of $\log x$. Furthermore, Lemma 1 can easily be extended to obtain from a bound on $|S(x^\phi, 2x^\phi)|$ consisting of a sum of several terms of the form $x^{\alpha\pm\beta\phi}$ or x^α a corresponding bound on $|S(x^{v_1}, 2x^{v_2})|$. To make use of the lemma, we will arrive at estimates for $|S(x^\phi, 2x^\phi)|$ for two different ranges of ϕ. To begin with, suppose that u and $u+a$ are in $S(x^\phi, 2x^\phi)$. Then there exist integers m_1 and m_2 such that $m_1 u^2 \in (x, x+h]$ and $m_2(u+a)^2 \in (x, x+h]$. Observe that since u and $u+a \in (x^\phi, 2x^\phi]$,

$$m_1 = \frac{x}{u^2} + O\left(\frac{h}{u^2}\right) = \frac{x}{u^2} + O\left(hx^{-2\phi}\right) \tag{1}$$

and

$$m_2 = \frac{x}{(u+a)^2} + O\left(hx^{-2\phi}\right). \tag{2}$$

Since u and $u+a \in S(x^\phi, 2x^\phi)$, we get that

(3)

$$\begin{aligned} m_1(2u-a) - m_2(2u+3a) &= \frac{x}{u^2}(2u-a) - \frac{x}{(u+a)^2}(2u+3a) + O\left(hx^{-\phi}\right) \\ &= \frac{x}{u^2(u+a)^2}\left((u+a)^2(2u-a) - u^2(2u+3a)\right) + O\left(hx^{-\phi}\right) \\ &= \frac{-a^3x}{u^2(u+a)^2} + O\left(hx^{-\phi}\right). \end{aligned}$$

The second of the three expressions in (3) we view as a difference for x/u^2 modified by the appearance of the polynomials $2u-a$ and $2u+3a$. If a is small in (3), then one gets that $m_1(2u-a)-m_2(2u+3a)=0$. The above idea is due to Roth [11] and, as observed by Nair [8], one can use (3) to show that if I is a subinterval of $(x^\phi, 2x^\phi]$ with $|I| \le x^{(4\phi-1)/3}/4$, then $|S(x^\phi, 2x^\phi) \cap I| \le 2$ (cf. [3]). This easily leads to

$$|S(x^\phi, 2x^\phi)| \ll x^{(1-\phi)/3} \qquad \text{for} \quad h\sqrt{\log x} < x^\phi \le 2\sqrt{x}. \tag{4}$$

Observe that (4) does not imply that $|S(x^\phi, 2x^\phi)| \ll x^{1/5}\log x$ when $\phi < 2/5$ (or, more specifically, when x^ϕ is of a smaller order than $x^{2/5}\log^{-3} x$). Thus, to obtain the theorem, we can only make use of (4) when $\phi \ge 2/5$. In the range $\phi \ge 2/5$, however, there is a better estimate for $|S(x^\phi, 2x^\phi)|$. The better estimate is

$$|S(x^\phi, 2x^\phi)| \ll x^{1-2\phi} \qquad \text{for} \quad x^{1/3} \le x^\phi \le 2\sqrt{x}. \tag{5}$$

Although either (4) or (5) will suffice for obtaining the theorem, we will use (5) in this paper for two reasons. First, as we have already mentioned, (5) is stronger than (4) when ϕ is restricted to the range $\phi \ge 2/5$. Thus, any future improvements on the theorem will more likely benefit from making use of (5) rather than (4). Also, the authors feel that (5) is easier to obtain than (4), and so it seems reasonable to use a simpler result (especially when it is also stronger). On the other hand, Roth's ideas will still play a very important role in proving the theorem. Although making use of (4) when $\phi \le 2/5$ does not help in obtaining the theorem, we will make much use of (3) when $\phi \le 2/5$.

We now turn to proving (5). First, we observe that a close examination of Davenport or Estermann's approach described in [11] easily leads to (5) (also see [6] or [13]). The authors' elementary approaches thus far have emphasized the use of differences, and so we show here how to view (5) as a consequence of differences. Observe that it suffices to show that if I is a subinterval of $(x^\phi, 2x^\phi]$ with $|I| \le (1/3)x^{3\phi-1}$, then $|S(x^\phi, 2x^\phi) \cap I| \le 1$. Suppose that there exist u and $u+a$ in $S(x^\phi, 2x^\phi) \cap I$ where $|I| \le (1/3)x^{3\phi-1}$. Defining m_1 and m_2 as above, we get that

$$m_1 - m_2 = \frac{x}{u^2} - \frac{x}{(u+a)^2} + O\left(hx^{-2\phi}\right) = \frac{a(2u+a)x}{u^2(u+a)^2} + O\left(hx^{-2\phi}\right).$$

Since u and $u+a$ are in I (and x is sufficiently large), the right-hand side above has absolute value < 1. On the other hand, it is easy to check that the first term on the right-hand side above is greater than the absolute value of the error term. This implies that $m_1 - m_2$ is an integer in $(0,1)$, giving a contradiction. Hence, we get the desired result.

Before finishing this section, we observe that an application of Lemma 1 and (5) give that

$$\left| S\left(x^{(1-\theta)/2}, 2\sqrt{x}\right)\right| \ll x^\theta$$

(where we can allow $\theta \le 1/5$ with $h = cx^\theta$ or $h = cx^\theta \log x$) so that, in particular,

$$\left| S\left(x^{2/5}, 2\sqrt{x}\right)\right| \ll x^{1/5}. \tag{6}$$

3. Second Differences

The methods discussed in the first section may be viewed as first difference techniques. To obtain (3) or (4) one uses a modified first difference of x/u^2. To obtain (5) one uses a direct application of a first difference. In this section, we make use of second differences. Our goal now is to find a good estimate for $S(x^\phi, 2x^\phi)$ when $\phi \leq 2/5$. Therefore, suppose now that ϕ satisfies this condition. We begin by supposing that u and $u+a$ are consecutive elements of $S(x^\phi, 2x^\phi)$ and that $a \geq x^{(4\phi-1)/3}/8$. Suppose further that $u+b$ and $u+b+a$ are also in $S(x^\phi, 2x^\phi)$ and that $b \geq a + (1/8)x^{(4\phi-1)/3}$. We view a as being fixed and show that there is a sufficiently small absolute positive constant c_1 such that any number b as above must also satisfy $b > c_1 a^{-1/3} x^{(5\phi-1)/3}$. This result is a consequence of the work of the first author in [1] where he uses second differences and the ideas of Halberstam and Roth [7] to obtain a result about gaps between $k-$free numbers. We will give a different approach here which is based on the use of divided differences, the basis of the second author's work in [14] and [15]. Both approaches are second difference approaches. For this particular application, there is a slight advantage to using a divided difference approach in that the error terms involved will be smaller than in the previous approach.

Let m_1, m_2, m_3, and m_4 be integers such that $m_1 u^2$, $m_2(u+a)^2$, $m_3(u+b)^2$, and $m_4(u+a+b)^2$ are all in $(x, x+h]$. If one uses (1), (2), and the corresponding expressions for m_3 and m_4, one can view $bm_1 - (a+b)m_2 + am_4$ and $am_1 - (a+b)m_3 + bm_4$ as divided differences for x/u^2. It is interesting to note that the first of these by itself can be used to obtain (4). (Also, a direct application of a third difference can be used to obtain (4).) We will use these two expressions for divided differences, however, in a different way. We observe that the first of these minus the second gives

$$(b-a)m_1 - (a+b)m_2 + (a+b)m_3 - (b-a)m_4$$

$$= (b-a)\frac{x}{u^2} - (a+b)\frac{x}{(u+a)^2} + (a+b)\frac{x}{(u+b)^2} - (b-a)\frac{x}{(u+a+b)^2} + O\left(\frac{(a+b)h}{u^2}\right)$$

$$= \frac{ab(b-a)(a+b)(2u+a+b)(2u^2+2au+2bu+ab)x}{u^2(u+a)^2(u+b)^2(u+a+b)^2} + O\left(\frac{(a+b)h}{u^2}\right).$$

Now, u and $u+a+b$ are both in $(x^\phi, 2x^\phi]$ so that $u > x^\phi > a+b$. In particular, this implies that if c_1 is sufficiently small and $b \leq c_1 a^{-1/3} x^{(5\phi-1)/3}$, then the first term on the right-hand side above has absolute value $< 1/2$. The conditions $a \geq x^{(4\phi-1)/3}/8$, $b \geq a + (1/8)x^{(4\phi-1)/3}$, and $x^\phi > h\sqrt{\log x}$ all imply that the first term on the right-hand side above is also greater than the absolute value of the error term above. Thus, we get that $(b-a)m_1 - (a+b)m_2 + (a+b)m_3 - (b-a)m_4$ is an integer $\in (0,1)$. Hence, we must have in fact that $b > c_1 a^{-1/3} x^{(5\phi-1)/3}$, as we had set out to establish.

4. The Use of Roth's Method

In this section, we make use of the idea of Roth given by equation (3) from section 2 and the result established in section 3 to complete the proof of the theorem. Let $R = x^{(4\phi-1)/3}/8$. Recall that in section 2, we mentioned that if I is a subinterval of $(x^\phi, 2x^\phi]$ with $|I| \leq 2R$, then $|S(x^\phi, 2x^\phi) \cap I| \leq 2$. Indeed, this is how one can establish (4). Although we will make use of this result, we do not include its proof here. For a proof, one can consult Roth's paper [11] as well as [3]. For ϕ fixed, define

$$T(a) = \{u : u \text{ and } u + a \text{ are consecutive elements in } S(x^\phi, 2x^\phi)\},$$

and

$$t(a) = |T(a)|.$$

Note that

$$|S(x^\phi, 2x^\phi)| \leq 1 + \sum_{a=1}^{\infty} t(a). \tag{7}$$

From the comments above, we know that of every 3 consecutive elements in $S(x^\phi, 2x^\phi)$, there exist 2 consecutive elements of distance $> R$ from one another. In other words, we get that

$$\sum_{a \leq R} t(a) \leq 1 + \sum_{a > R} t(a).$$

Hence, from (7), we get that

$$|S(x^\phi, 2x^\phi)| \leq 2 + 2\sum_{a > R} t(a). \tag{8}$$

Let $a > R$. We now bound $t(a)$ from above. To do this, we consider u and $u + b$ as two non-consecutive elements in $T(a)$. In particular, this implies that $u + b \geq u + 2a$ so that $b \geq a + R$. Hence, the conditions of section 3 hold, and we get that

$$b > c_1 a^{-1/3} x^{(5\phi-1)/3}. \tag{9}$$

We now consider a subinterval I of $S(x^\phi, 2x^\phi)$ with $|I| \leq c_2 a^{-3} x^{5\phi-1}$ where c_2 is a sufficiently small positive constant to be chosen momentarily. We will show that the number of $u \in T(a) \cap I$ is $\ll c a^{-8/3} x^{(7\phi+3\theta-2)/3} \log x + 1$. Observe that since $a > R$, we get that $|I| \leq c_2 a^{-3} x^{5\phi-1} \leq x^\phi$ (provided $c_2 \leq 1/512$). Thus, once the above bound on $|T(a) \cap I|$ is established, we will have that

$$\begin{aligned} t(a) &\ll \frac{x^\phi}{a^{-3} x^{5\phi-1}} \left(c a^{-8/3} x^{(7\phi+3\theta-2)/3} \log x + 1 \right) \\ &= c a^{1/3} x^{(-5\phi+3\theta+1)/3} \log x + a^3 x^{1-4\phi}. \end{aligned} \tag{10}$$

Suppose that u and $u+b \in T(a) \cap I$. Then there exist integers m_1, m_2, m_3, and m_4 such that $m_1 u^2$, $m_2(u+a)^2$, $m_3(u+b)^2$, and $m_4(u+a+b)^2$ are all in $(x, x+h]$. From (3), we get that

$$m_1(2u-a) - m_2(2u+3a) = \frac{-a^3 x}{u^2(u+a)^2} + O\left(hx^{-\phi}\right)$$

and

$$m_3(2u+2b-a) - m_4(2u+2b+3a) = \frac{-a^3 x}{(u+b)^2(u+a+b)^2} + O\left(hx^{-\phi}\right).$$

A simple calculation now gives that

$$\begin{aligned}
&m_3(2u+2b-a) - m_4(2u+2b+3a) - m_1(2u-a) + m_2(2u+3a)\\
&= \frac{a^3 x}{u^2(u+a)^2(u+b)^2(u+a+b)^2}\left((u+b)^2(u+a+b)^2 - u^2(u+a)^2\right) + O\left(hx^{-\phi}\right)\\
&= \frac{a^3 x}{u^2(u+a)^2(u+b)^2(u+a+b)^2} b(2u+a+b)\left(2u^2+2au+2bu+ab+b^2\right) + O\left(hx^{-\phi}\right).
\end{aligned}$$

Since both u and $u+b$ are in I, we get that $b \le |I| \le c_2 a^{-3} x^{5\phi-1}$. Thus, if c_2 is sufficiently small, we get that the first term on the right-hand side of the equation above is $< 1/2$. We choose $c_2 > 0$ so that the above all holds. Now, there is a sufficiently large constant c_3 such that if $b > c_3 a^{-3} x^{4\phi-1} h$, then the first term on the right-hand side above is greater than the absolute value of the error term. But this would make the right-hand side above strictly between 0 and 1 which is an impossibility since $m_3(2u+2b-a) - m_4(2u+2b+3a) - m_1(2u-a) + m_2(2u+3a)$ is an integer. Thus, in fact, we must have that

$$b \le c_3 a^{-3} x^{4\phi-1} h = c_3 c a^{-3} x^{4\phi+\theta-1} \log x.$$

Observe that this bound on b is an upper bound on the distance between any 2 elements of $T(a) \cap I$. On the other hand, (9) provides us with a lower bound on the distance between any 2 non-consecutive elements of $T(a) \cap I$. Hence, we get that

$$|T(a) \cap I| \le 2\frac{c_3 c a^{-3} x^{4\phi+\theta-1} \log x}{c_1 a^{-1/3} x^{(5\phi-1)/3}} + 2 \ll c a^{-8/3} x^{(7\phi+3\theta-2)/3} \log x + 1.$$

This was the bound on $|T(a) \cap I|$ that we wanted; hence, (10) holds. Let $B = x^{(5\phi-1)/5}$. From (10) we get that

$$\begin{aligned}
\sum_{R<a\le B} t(a) &\ll \sum_{R<a\le B} \left(c a^{1/3} x^{(-5\phi+3\theta+1)/3} \log x + a^3 x^{1-4\phi}\right)\\
&\ll c B^{4/3} x^{(-5\phi+3\theta+1)/3} \log x + B^4 x^{1-4\phi}\\
&= c x^{(-5\phi+15\theta+1)/15} \log x + x^{1/5}.
\end{aligned}$$

Also,

$$x^{\phi} \geq \sum_{a=1}^{\infty} at(a) \geq \sum_{a>B} at(a) \geq B \sum_{a>B} t(a)$$

so that

$$\sum_{a>B} t(a) \leq \frac{x^{\phi}}{B} = x^{1/5}.$$

Combining the above with (8), we now get that

$$|S(x^{\phi}, 2x^{\phi})| \ll cx^{(-5\phi+15\theta+1)/15} \log x + x^{1/5} \qquad \text{for } \theta < \phi \leq 2/5.$$

We now make use of Lemma 1, as modified by the comments following it, to obtain that

$$|S(h\sqrt{\log x}, x^{2/5})| \ll c\left(h\sqrt{\log x}\right)^{-1/3} x^{(15\theta+1)/15} \log x + x^{1/5} \log x$$

$$\ll c^{2/3} x^{1/5} (\log x)^{1/2} + x^{1/5} \log x.$$

Combining this with (6) gives that

$$|S(h\sqrt{\log x}, 2\sqrt{x})| \ll c^{2/3} x^{1/5} (\log x)^{1/2} + x^{1/5} \log x \ll c^{2/3} x^{1/5} \log x,$$

which completes the proof.

In conclusion, the authors thank Vasil Popov for initiating the collaboration which has led to the authors' joint work on the gap problem for squarefree numbers.

References

1. M. Filaseta, *An elementary approach to short intervals results for k–free numbers*, J. Number Theory **30** (1988), 208–225.
2. M. Filaseta, *Short interval results for squarefree numbers*, J. Number Theory **35** (1990), 128–149.
3. M. Filaseta and O. Trifonov, *On gaps between squarefree numbers*, Analytic Number Theory, Proceedings of a Conference in Honor of Paul T. Bateman (Progress in Mathematics Series, Vol. 85), edited by Berndt, Diamond, Halberstam, and Hildebrand, Birkhäuser, Boston (1990), 235–253.
4. E. Fogels, *On the average values of arithmetic functions*, Proc. Cambridge Philos. Soc. **37** (1941), 358–372.
5. S. W. Graham and G. Kolesnik, *On the difference between consecutive squarefree integers*, Acta Arith. **49** (1988), 435–447.
6. H. Halberstam, *Gaps in integer sequences*, Math. Mag. **56** (1983), 131–140.
7. H. Halberstam and K. F. Roth, *On the gaps between consecutive k–free integers*, J. London Math. Soc. (2) **26** (1951), 268–273.
8. M. Nair, *Power free values of polynomials II*, Proc. London Math. Soc. (3) **38** (1979), 353–368.

9. R. A. Rankin, *Van der Corput's method and the theory of exponent pairs*, Quart. J. Math. Oxford Ser. (2) **6** (1955), 147–153.
10. H. E. Richert, *On the difference between consecutive squarefree numbers*, J. London Math. Soc. (2) **29** (1954), 16–20.
11. K. F. Roth, *On the gaps between squarefree numbers*, J. London Math. Soc. (2) **26** (1951), 263–268.
12. P. G. Schmidt, *Abschätzungen bei unsymmetrischen Gitterpunktproblemen*, Dissertation zur Erlangung des Doktorgrades der Mathematisch–Naturwissenschaftlichen Fakultät der Georg–AugustUniversität zu Göttingen, 1964.
13. H. N. Shapiro, *Introduction to the Theory of Numbers*, John Wiley & Sons, New York, 1983, pp. 294295.
14. O. Trifonov, *On the squarefree problem*, C. R. Acad. Bulgare Sci. **41** (1988), 37–40.
15. O. Trifonov, *On the squarefree problem II*, Mathematica Balcanika **3** (1989), 284–295.

Mathematics Department
University of South Carolina
Columbia, SC 29208

Center of Mathematics and Mechanics
Bulgarian Academy of Sciences
1090 Sofia, P.O. Box 373
Bulgaria