SHORT INTERVAL RESULTS FOR k–FREE VALUES OF IRREDUCIBLE POLYNOMIALS

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Dedicated to the memory of a friend, David R. Richman

1. Introduction

Let k be an integer ≥ 2. Trifonov and the author (cf. [3],[4],[15]) have recently made improvements on the gap problem of finding an $h = h(x)$ as small as possible such that for x sufficiently large, every interval of the form $(x, x+h]$ contains a k–free number. Although, one expects such h to exist with h as small as $c \log^2 x$ for some constant c (K. McCurley and A. Zaccagnini, independent private communications), the best result to date is that one can take $h = cx^{1/(2k+1)} \log x$. This was established recently in the case $k = 2$ by Trifonov and the author [4] and for general k by Trifonov [15]. One can generalize this problem by considering an irreducible polynomial $f(z) \in \mathbb{Z}[z]$ and ask for an $h = h(f(z), x)$ as small as possible such that for x sufficiently large, there is an $m \in (x, x+h]$ for which $f(m)$ is k–free. Necessarily, one needs to require that

$$\gcd(f(m), m \in \mathbb{Z}) \text{ is } k-\text{free}. \tag{1}$$

The gap problem mentioned above then corresponds to the case that $f(z) = z$. The general problem was considered by Cugiani [1] and Nair [9,10] and is related to work of Nagel [8], Ricci [12], Erdős [2], Hooley [6], and Huxley and Nair [7]. In particular, Nair [10] showed that if $k \geq n+1$ where n is the degree of $f(z)$, then one can take

$$h = cx^{n/(2k-n+1)}. \tag{2}$$

This result generalized (and slightly improved) a theorem of Halberstam and Roth [5] which stated that for every $\epsilon > 0$ and for $h = x^{1/(2k)+\epsilon}$, there is a k–free number in the interval $(x, x+h]$. We note that if $k \geq n+1$ and the greatest common divisor of the coefficients of $f(z)$ is 1, then (1) holds (cf. [11]). An improvement on (2) follows from the work of Huxley and Nair [7 (take $t = k - g + 1$ in Theorem A)]. Their work implies that if $k \geq n+1 \geq 3$, then one can take

$$h = cx^{n/(2k-n+2)}. \tag{3}$$

One can further reduce h by a power of $\log x$. A direct application of the techniques in [15] do not improve on (3). The purpose of this paper is to show that one can in general do significantly better than (3) by employing different methods. Our methods here will be based on establishing some polynomial identities which are reminiscent of polynomial identities used by Huxley and Nair [7]. We shall also make use of divided differences which were initially used for such problems in the work of Trifonov [13,14] (also, see [3]). Our result is

Theorem. *Let $f(z) \in \mathbb{Z}[z]$ of degree n with $f(z)$ irreducible. Let k be an integer $\geq n+1$ satisfying (1). Let r be the greatest integer satisfying $r(r-1) < 2n$. Then there is a constant $c = c(f(z), k)$ such that for x sufficiently large, there is an integer $m \in (x, x+h]$ for which $f(m)$ is $k-$free where*

$$h = cx^{n/(2k-n+r)}.$$

Observe that since $r(r+1) \geq 2n$, one may replace r above with $(-1+\sqrt{8n+1})/2$ or with $\sqrt{2n} - (1/2)$. Thus, we have increased the denominator in (3) by $> \sqrt{2n} - (5/2)$. One can again reduce h by a power of $\log x$ in our results, but we do not concern ourselves with this reduction. We note that with h as in the Theorem, one can obtain asymptotics for the number of $m \in (x, x+h]$ for which $f(m)$ is $k-$free. Asymptotic results were obtained by the previous authors in their works. Also, we do not consider the case that $k \leq n$; however, the material in this paper is presented in such a way that the techniques can easily be applied to such k. Given an irreducible polynomial $f(z)$ of degree n, Nair [9] obtained estimates for the smallest k such that if (1) holds, then there are infinitely many integers m for which $f(m)$ is $k-$free. To obtain his results, he necessarily considered $k \leq n$. The results of Huxley and Nair [7] give a slight improvement on that work. More specifically, they show that for some values of n, one can reduce the smallest k permissible in the work of Nair [9,10] by 1. Our methods do not improve further on this application of the results in [7], so we do not emphasize results related to $k \leq n$. We note, however, that some improvement on the gap problem for $k \leq n$ can easily be obtained from the methods here and that the larger k is the better the resulting improvement.

2. Preliminaries

We will make use of the following notation:

$f(z)$ is an irreducible polynomial in $\mathbb{Z}[z]$.
n is the degree of $f(z)$.
k and m will denote positive rational integers with k satisfying (1).
x is a sufficiently large real number (depending on $f(z)$ and k).
$h = h(k, f(z), x)$ is such that $\lim_{x\to\infty} h = \infty$.
μ is a fixed root of $f(z)$.
$K = \mathbb{Q}(\mu)$.
R is the ring of integers in K.
$\{\omega_1, \dots, \omega_n\}$ is a fixed integral basis for K over $\mathbb{Q}$.
$\sigma_1, \dots, \sigma_n$ denote the homomorphisms of K which fix the elements of $\mathbb{Q}$.
E will be a fixed element in R.

$N(u) = N_{K/\mathbb{Q}}(u) = \prod_{j=1}^{n} \sigma_j(u)$ (where $u \in K$).

$||u||$ denotes the size of an element u in K ($||u|| = \max_{1\le j\le n} |\sigma_j(u)|$).

$c, c_1, c_2 \ldots$ and implied constants, unless otherwise stated, are positive constants depending on $f(z)$ and k. Constants other than c are independent of c.

u is *primary* means that $||u|| < c_1|N(u)|^{1/n}$ where c_1 is a constant (cf. [9]). This differs slightly from Nair's use of the word "primary," but it is sufficient for obtaining our results.

J denotes a subinterval of $(x, x+h]$.

S_J denotes the set of $u \in R$ such that u is primary and such that there is a $v = v(u) \in R$ and a rational integer $m = m(u) \in J$ for which $u^k v = E(m-\mu)$.

$S_J(a,b) = \{u \in S_J : a^{1/n} < ||u|| \le b^{1/n}\}$.

$S(a,b) = S_{(x,x+h]}(a,b)$.

Lemma 1. *Let $T > 0$. Let*

$$N_k(x) = \left|\{m : x < m \le x+h,\ f(m) \text{ is } k-\text{free}\}\right|,$$

$$P(x) = \left|\left\{m : x < m \le x+h,\ p^k|f(m) \text{ for some prime } p > T\right\}\right|,$$

and

$$\rho(p^k) = \left|\left\{j \in \{0,1,\ldots,p^k-1\} : f(j) \equiv 0 \pmod{p^k}\right\}\right|.$$

Then

$$N_k(x) = h\prod_p \left(1 - \frac{\rho(p^k)}{p^k}\right) + O\left(\frac{h}{(\log x)^{k-1}}\right) + O(\pi(T)) + O(P(x))$$

and

$$P(x) \ll \max \sum 1,$$

where the maximum is over all E from a fixed finite set of algebraic integers in K and the sum is over all pairs (u,v) with u and $v \in R$, u primary, $||u|| > c_2 T^{1/n}$, and $u^k v = E(m-\mu)$ for some rational integer $m \in (x, x+h]$.

The proof of Lemma 1 can be found in [9], so we omit its proof here. We take $T = h\sqrt{\log x}$. Thus, the error term involving $\pi(T)$, which represents the number of primes $\le T$, is $\ll h/\sqrt{\log x}$. To estimate $P(x)$, we divide the interval $(x, x+h]$ into subintervals of length H. Our goal will be to find an upper bound on $S_J(t, 2^n t)$, say U, which is independent of the subinterval $J \subseteq (x, x+h]$ with $|J| = H$ and independent of E. It will then follow that

$$|S(t, 2^n t)| \ll \left(\frac{h}{H} + 1\right) U. \tag{3}$$

Then we will use that

$$P(x) \ll \sum_{j=0}^{\infty} \left|S(2^{jn} c_2^n T, 2^{(j+1)n} c_2^n T)\right|.$$

This idea for bounding $P(x)$ can be found in [7], [9], or [10].

Our next lemma provides us with the means to estimate $S_J(t, 2^n t)$ and, hence, the right-hand side of (3). Again the result is a consequence of [9].

Lemma 2. *Let $J \subseteq (x, x+h]$. Let $B > 0$ and $t > 0$. Suppose that for every $u \in S_J(t, 2^n t)$, there are $\ll 1$ numbers α with $||\alpha|| \le B$ for which $u + \alpha \in S_J(t, 2^n t)$. Then*

$$|S_J(t, 2^n t)| \ll \frac{t}{B^n} + 1.$$

With s an integer in $[1, k-1]$ and with

$$H = c_3 t^{(k-s)/n}$$

for some constant c_3, Nair [10] showed that one can take

$$B = c_4 x^{-1/(2s+1)} t^{(k+s+1)/(n(2s+1))}, \tag{4}$$

for some constant c_4. Huxley and Nair [7] obtained improvements on the results in [10] by showing that Nair's choice of B above can be used with

$$H = c_5 t^{2s(k-s)/(n(2s+1))} x^{1/(2s+1)},$$

for some c_5. In this paper, we will pursue the ideas of Huxley and Nair a little further and show that if we decrease Nair's choice of B by a small amount, we can increase his choice of H by a considerably larger amount. Their work was based on constructing polynomials with some good approximation properties, and likewise we will need to develop similar polynomials.

3. Some Polynomial Identities

The work of Halberstam and Roth [5] was based on a particular polynomial identity which was later generalized by Nair [9]. The polynomials which occurred in Nair [9] were not given explicitly until the work of Huxley and Nair [7]. The following lemma follows from the latter (though our polynomial $Q_{s,k}(u, \alpha)$ is expressed differently).

Lemma 3. *Let s be a non-negative integer $\le k-1$. Let*

$$P_{s,k}(u, \alpha) = \frac{(k+s)!}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} \frac{(2s-j)!}{(k+s-j)!} \alpha^j (u+\alpha)^{s-j}$$

and

$$Q_{s,k}(u, \alpha) = \frac{(k+s)!}{s!} \sum_{j=0}^{s} \binom{s}{j} \frac{(2s-j)!}{(k+s-j)!} \alpha^j u^{s-j}.$$

Then $P_{s,k}(u, \alpha)$ and $Q_{s,k}(u, \alpha)$ are homogeneous polynomials in $\mathbb{Z}[u, \alpha]$ of degree s which satisfy

$$(u+\alpha)^k P_{s,k}(u, \alpha) - u^k Q_{s,k}(u, \alpha) = G_{s,k}(u, \alpha), \tag{5}$$

where $G_{s,k}(u,\alpha)$ is a polynomial of degree $k-s-1$ in the variable u and, hence, divisible by α^{2s+1}.

The polynomials in Lemma 3 will play a major role in the arguments of this paper; we, therefore, present a proof of Lemma 3. Our proof will differ from that given by Huxley and Nair. It is easy to verify that $P_{s,k}(u,\alpha)$ and $Q_{s,k}(u,\alpha)$ are homogeneous polynomials in $\mathbb{Z}[u,\alpha]$ of degree s, so we only concern ourselves with establishing (5). For convenience, we ignore for the moment concerns about our polynomials being in $\mathbb{Z}[u,\alpha]$ and seek first to construct $P_{s,k}(u,\alpha)$ and $Q_{s,k}(u,\alpha)$ in $\mathbb{Q}[u,\alpha]$ which are homogeneous polynomials of degree s and which satisfy (5). To motivate the argument, we assume first that we have polynomials satisfying (5). By differentiating the equation

$$(u+\alpha)^{k+1}P_{s,k+1}(u,\alpha) - u^{k+1}Q_{s,k+1}(u,\alpha) = G_{s,k+1}(u,\alpha)$$

with respect to the variable u, we obtain (5) with

$$P_{s,k}(u,\alpha) = (k+1)P_{s,k+1}(u,\alpha) + (u+\alpha)P'_{s,k+1}(u,\alpha),$$

$$Q_{s,k}(u,\alpha) = (k+1)Q_{s,k+1}(u,\alpha) + uQ'_{s,k+1}(u,\alpha),$$

and

$$G_{s,k}(u,\alpha) = G'_{s,k+1}(u,\alpha).$$

In other words,

$$\frac{d}{du}\left((u+\alpha)^{k+1}P_{s,k+1}(u,\alpha)\right) = (u+\alpha)^{k}P_{s,k}(u,\alpha),$$

and

$$\frac{d}{du}\left(u^{k+1}Q_{s,k+1}(u,\alpha)\right) = u^{k}Q_{s,k}(u,\alpha).$$

Hence, we want

$$P_{s,k+1}(u,\alpha) = \frac{1}{(u+\alpha)^{k+1}}\int (u+\alpha)^{k}P_{s,k}(u,\alpha)\,du \tag{6}$$

and

$$Q_{s,k+1}(u,\alpha) = \frac{1}{u^{k+1}}\int u^{k}Q_{s,k}(u,\alpha)\,du, \tag{7}$$

where the constants of integration (which may depend on α) are chosen so that the right-hand sides above are homogeneous polynomials in $\mathbb{Q}[u,\alpha]$. We are not making claims yet that the constants can be so chosen. The point here is that if an identity like (5) is to be possible, then the above must all be possible. We are therefore motivated to use (6) and (7) to construct our polynomials.

We have not yet defined $P_{k,k}(u,\alpha)$ and $Q_{k,k}(u,\alpha)$, but it is convenient to do so. We define

$$P_{k,k}(u,\alpha) = u^{k} \quad \text{and} \quad Q_{k,k}(u,\alpha) = (u+\alpha)^{k}.$$

Thus,

$$(u+\alpha)^k P_{k,k}(u,\alpha) - u^k Q_{k,k}(u,\alpha) = 0. \tag{8}$$

Motivated by the above, we integrate both sides of this equation with respect to u. Observe that with the change of variable $v = u+\alpha$ and a suitable choice of constants of integration

$$\begin{aligned}\int (u+\alpha)^k P_{k,k}(u,\alpha)\,du &= \int (u+\alpha)^k u^k\,du = \int v^k (v-\alpha)^k\,dv \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \alpha^j \int v^{2k-j}\,dv = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{2k-j+1} \alpha^j (u+\alpha)^{2k-j+1}.\end{aligned}$$

This last expression is divisible by $(u+\alpha)^{k+1}$ so that after integrating the first term in (8), we can rewrite it in the form $(u+\alpha)^{k+1}P_{k,k+1}(u,\alpha)$ for some $P_{k,k+1}(u,\alpha) \in \mathbb{Q}[u,\alpha]$. Similarly, after integration, we can rewrite the second term in (8) in the form $u^{k+1}Q_{k,k+1}(u,\alpha)$. In other words, after integrating in (8) and replacing k with $k-1$, we are led to

$$(u+\alpha)^k P_{k-1,k}(u,\alpha) - u^k Q_{k-1,k}(u,\alpha) = G_{k-1,k}(u,\alpha), \tag{9}$$

where

$$P_{k-1,k}(u,\alpha) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{1}{2k-j-1} \alpha^j (u+\alpha)^{k-j-1},$$

$$Q_{k-1,k}(u,\alpha) = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{1}{2k-j-1} \alpha^j u^{k-j-1},$$

and $G_{k-1,k}(u,\alpha)$ is necessarily a polynomial depending only on α. It is not difficult to determine $G_{k-1,k}(u,\alpha)$ explicitly, but observe that since $P_{k-1,k}(u,\alpha)$ is of degree $k-1$ in u, it is not divisible by u^k so that (9) implies that $G_{k-1,k}(u,\alpha) \not\equiv 0$. Also, the left-hand side of (9) is a homogeneous polynomial of degree $2k-1$ so that $G_{k-1,k}(u,\alpha)$ must be also.

We now continue by repeatedly integrating both sides of (9). It is easy to check that after a total of $k-s$ integrations (replacing k by $k-1$ after each integration), one is led to (5) with

$$P_{s,k}(u,\alpha) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} \frac{1}{(2s-j+1)(2s-j+2)\cdots(2s-j+(k-s))} \alpha^j (u+\alpha)^{s-j}$$

and

$$Q_{s,k}(u,\alpha) = \sum_{j=0}^{s} \binom{s}{j} \frac{1}{(2s-j+1)(2s-j+2)\cdots(2s-j+(k-s))} \alpha^j u^{s-j}.$$

The polynomials thus constructed are in $\mathbb{Q}[u,\alpha]$; after multiplication by $\dfrac{(k+s)!}{s!}$, we are left with polynomials in $\mathbb{Z}[u,\alpha]$. The resulting polynomials can easily be rewritten in the form given by the lemma.

Corollary. *Let $P_{s,k}(u,\alpha)$ and $Q_{s,k}(u,\alpha)$ be as in Lemma 3. Then*

$$Q_{s,k}(u,\alpha) = P_{s,k}(u+\alpha, -\alpha).$$

This simple Corollary will be useful in the remainder of this section. It is also motivated by the fact that if one replaces α with $-\alpha$ and then u with $u+\alpha$ in (5), then one is left with an equation which is similar to (5).

The polynomials $P_{s,k}(u,\alpha)$ given by Lemma 3 here are the same as those obtained from Lemma 2 of [7] with $e = f = s$ and $x = \alpha/(u+\alpha)$. Observe that (5) implies that for $j \in \{0, 1, \dots, s\}$, the coefficient of $\alpha^{s-j}u^{k+j}$ in $(u+\alpha)^k P_{s,k}(u,\alpha)$ is the same as the coefficient of $\alpha^{s-j}u^j$ in $Q_{s,k}(u,\alpha)$. In particular, (5) and $P_{s,k}(u,\alpha)$ uniquely determine $Q_{s,k}(u,\alpha)$. This implies that if we obtain $Q_{s,k}(u,\alpha)$ from Lemma 2 of [7] in the same manner that we obtained $P_{s,k}(u,\alpha)$, then we must get the same $Q_{s,k}(u,\alpha)$ given in Lemma 3. Hence, we get from the work of Huxley and Nair [7] that

$$Q_{s,k}(u,\alpha) = \sum_{j=0}^{s} \frac{(k-s+j-1)!}{(k-s-1)!} \cdot \frac{(2s-j)!}{s!} \binom{s}{j} \alpha^j (u+\alpha)^{s-j}.$$

The Corollary of Lemma 3 implies that $P_{s,k}(u,\alpha) = Q_{s,k}(u+\alpha,-\alpha)$. Thus, we obtain the following new expression for $P_{s,k}(u,\alpha)$.

Lemma 4.

$$P_{s,k}(u,\alpha) = \sum_{j=0}^{s} (-1)^j \frac{(k-s+j-1)!}{(k-s-1)!} \cdot \frac{(2s-j)!}{s!} \binom{s}{j} \alpha^j u^{s-j}.$$

Before continuing, we give a brief description of what our immediate goal is. For the moment, fix s as in Lemma 3. Let r be a positive integer. We seek next to find $r+1$ polynomials $P_0, \dots, P_r$ in $\mathbb{Z}[u, \alpha_1, \dots, \alpha_r]$ which cause the expression

$$\frac{P_0}{u^k} + \frac{P_1}{(u+\alpha_1)^k} + \cdots + \frac{P_r}{(u+\alpha_1+\cdots+\alpha_r)^k} \tag{10}$$

to be small in absolute value. One of course can find such P_j which cause the expression to be 0. In addition, however, we will want to choose the P_j so that at least one of them is non-zero and so that they are each themselves fairly small in absolute value. We will also wish to view u as being large in comparison to the α_j. Lemma 3 corresponds to the case in which $r = 1$ with $P_0 = P_{s,k}(u,\alpha_1)$ and $P_1 = -Q_{s,k}(u,\alpha_1)$. It was observed by Huxley and Nair [7] that by increasing the size of r to 2, it is possible to decrease the maximum size of the P_j in Lemma 3 without altering the relative size of the expression in (10). The main idea in this paper is centered around this idea of Huxley and Nair. We will choose r to be considerably larger in order to decrease the maximum size of the P_j in (10). We will do this at the cost of increasing the size of the expression in (10), but the factor we will be increasing this expression by will be considerably smaller than the factor we will be decreasing the size of the P_j by. This latter fact will enable us to get the results mentioned in the Introduction.

For simplicity in notation, we momentarily fix s and define

$$P(u,\alpha) = P_{s,k}(u,\alpha) \quad \text{and} \quad Q(u,\alpha) = Q_{s,k}(u,\alpha).$$

We consider the following array:

$$\begin{array}{llll}
P(u,\alpha_1) & P(u,\alpha_1+\alpha_2) & P(u,\alpha_1+\alpha_2+\alpha_3) & \cdots \\
P(u+\alpha_1,-\alpha_1) & P(u+\alpha_1,\alpha_2) & P(u+\alpha_1,\alpha_2+\alpha_3) & \cdots \\
P(u+\alpha_1+\alpha_2,-\alpha_1-\alpha_2) & P(u+\alpha_1+\alpha_2,-\alpha_2) & P(u+\alpha_1+\alpha_2,\alpha_3) & \cdots \\
P(u+\alpha_1+\alpha_2+\alpha_3,-\alpha_1-\alpha_2-\alpha_3) & P(u+\alpha_1+\alpha_2+\alpha_3,-\alpha_2-\alpha_3) & P(u+\alpha_1+\alpha_2+\alpha_3,-\alpha_3) & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

so that if $f_{i,j}$ is the element in the i^{th} row and j^{th} column, then

$$f_{i,j} = \begin{cases} P\left(u + \sum_{\ell=1}^{i-1} \alpha_\ell,\ -\sum_{\ell=j}^{i-1} \alpha_\ell\right) & \text{if } j \le i-1 \\ P\left(u + \sum_{\ell=1}^{i-1} \alpha_\ell,\ \sum_{\ell=i}^{j} \alpha_\ell\right) & \text{if } j \ge i. \end{cases} \tag{11}$$

We form our choice of polynomials P_i in (10) by considering a divided difference to approximate an $(r-1)^{\text{th}}$ derivative of $\pm P(u,\alpha)$ with respect to α. To obtain P_i, we make use of elements from the $(i+1)^{\text{th}}$ row above and the first r columns.

As an example, we consider the case $r=2$. We take

$$P_0 = f_{1,2} - f_{1,1},\ \ P_1 = -\left(f_{2,2} - f_{2,1}\right),\ \text{and}\ P_2 = f_{3,2} - f_{3,1}.$$

By the definition of $f_{i,j}$ and the Corollary to Lemma 3, we get in this case that

$$P_0 = P(u,\alpha_1+\alpha_2) - P(u,\alpha_1), \quad P_1 = -P(u+\alpha_1,\alpha_2) + Q(u,\alpha_1),$$

and

$$P_2 = Q(u+\alpha_1,\alpha_2) - Q(u,\alpha_1+\alpha_2).$$

Observe that $P(u,\alpha)$ and $Q(u,\alpha)$ are polynomials of total degree s in u and α and of degree s in the variable u whereas P_0, P_1, and P_2 are of total degree s in u and α but only of degree $s-1$ in u. If we clear the denominator in (10) and rearrange terms, we obtain

$$\begin{aligned}
&(u+\alpha_1)^k \left((u+\alpha_1+\alpha_2)^k P(u,\alpha_1+\alpha_2) - u^k Q(u,\alpha_1+\alpha_2)\right) \\
&\qquad - (u+\alpha_1+\alpha_2)^k \left((u+\alpha_1)^k P(u,\alpha_1) - u^k Q(u,\alpha_1)\right) \\
&\qquad - u^k \left((u+\alpha_1+\alpha_2)^k P(u+\alpha_1,\alpha_2) - (u+\alpha_1)^k Q(u+\alpha_1,\alpha_2)\right).
\end{aligned}$$

By Lemma 3, the above is a polynomial of degree at most $2k-s-1$ in the variable u. Viewing α_1 and α_2 as small compared to u, we get that the expression in (10) is of order

$\ll \max\{|\alpha_1|, |\alpha_2|\}^{2s+1} u^{-(k+s+1)}$. With $r = 1$, $P_0 = P_{s,k}(u,\alpha)$, and $P_1 = -Q_{s,k}(u,\alpha)$, Lemma 3 gives that (10) is $\ll |\alpha_1|^{2s+1} u^{-(k+s+1)}$. We will not distinguish yet between the relative sizes of the $|\alpha_j|$ so that the bounds we get on the expression in (10) for the cases that $r = 1$ and $r = 2$ are the same. Thus, we have decreased the maximum size of the P_j without altering the bounds we obtain for the expression in (10). This corresponds then to the role of the polynomials constructed by Huxley and Nair in [7, Lemma 6].

For general r, we proceed as follows. Consider i and j with $1 \le i \le r+1$ and $1 \le j \le r$. Set

$$\alpha'_\ell = \begin{cases} \alpha_\ell & \text{if } 1 \le \ell \le i-2 \\ \alpha_\ell + \alpha_{\ell+1} & \text{if } \ell = i-1 \\ \alpha_{\ell+1} & \text{if } i \le \ell \le r-1, \end{cases}$$

and

$$\alpha''_\ell = \begin{cases} \alpha'_\ell & \text{if } 1 \le \ell \le j-2 \\ \alpha'_\ell + \alpha'_{\ell+1} & \text{if } \ell = j-1 \\ \alpha'_{\ell+1} & \text{if } j \le \ell \le r-2. \end{cases}$$

Define

$$A_{i,j} = A^{(r)}_{i,j} = \prod_{1 \le \ell_1 \le \ell_2 \le r-2} \left(\alpha''_{\ell_1} + \alpha''_{\ell_1+1} + \cdots + \alpha''_{\ell_2} \right),$$

where the superscript will be used for later purposes. In the next section, we will also need to distinquish between different values of $A_{i,j}$ obtained from different choices of $\alpha_1, \ldots, \alpha_r$; we will write $A_{i,j} = A_{i,j}(\alpha_1, \ldots, \alpha_r)$ for such purposes. One easily checks that

$$A_{i,j} = A_{j+1,i} \quad \text{if } i \le j \qquad \text{and} \qquad A_{i,j} = A_{j,i-1} \quad \text{if } i > j. \tag{12}$$

Let

$$P_{i-1} = (-1)^{i-1} \sum_{j=1}^{r} (-1)^j A_{i,j} f_{i,j}. \tag{13}$$

The definitions of $P_0, \ldots, P_r$ given by (13) are simply divided differences of $\pm P(u,\alpha)$ which approximate the $(r-1)^{\text{th}}$ partial derivative of $\pm P(u,\alpha)$ with respect to α. There is an aspect of these definitions which is very important to us. Although the $(r-1)^{\text{th}}$ derivatives are taken with respect to α, the degreee of the polynomial with respect to u is decreased by $r-1$. To see this, observe that as a polynomial in u, the coefficient of u^{s-j} in $P(u,\alpha)$ is a polynomial of degree $\le j$ in α; in fact, it is a multiple of α^j. Hence, the divided difference above will result in the coefficients of $u^s, \ldots, u^{s-r+2}$ being 0. In other words, each P_i is a polynomial of degree at most $s-r+1$ in u.

From (13), we get that

$$\sum_{i=1}^{r+1} \frac{P_{i-1}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k}$$
$$= \sum_{i=1}^{r+1} \sum_{j=i}^{r} (-1)^{i+j-1} \frac{A_{i,j} f_{i,j}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k} + \sum_{i=1}^{r+1} \sum_{j=1}^{i-1} (-1)^{i+j-1} \frac{A_{i,j} f_{i,j}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k}.$$

Observe that in the first double sum on the right-hand side above, the range on i may be restricted to $1 \le i \le r$ since when $i = r+1$ the inner sum is vacuously 0. Also, (12) implies that

$$\begin{aligned}\sum_{i=1}^{r+1}\sum_{j=1}^{i-1}(-1)^{i+j-1}&\frac{A_{i,j}f_{i,j}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k}\\ &= \sum_{j=1}^{r}\sum_{i=j+1}^{r+1}(-1)^{i+j-1}\frac{A_{j,i-1}f_{i,j}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k}\\ &= \sum_{j=1}^{r}\sum_{i=j}^{r}(-1)^{i+j}\frac{A_{j,i}f_{i+1,j}}{(u+\alpha_1+\cdots+\alpha_i)^k}\\ &= \sum_{i=1}^{r}\sum_{j=i}^{r}(-1)^{i+j}\frac{A_{i,j}f_{j+1,i}}{(u+\alpha_1+\cdots+\alpha_j)^k}.\end{aligned}$$

From the Corollary to Lemma 3 and (11), we now get that

(14)
$$\begin{aligned}\sum_{i=1}^{r+1}&\frac{P_{i-1}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k}\\ &= \sum_{i=1}^{r}\sum_{j=i}^{r}(-1)^{i+j-1}\left(\frac{A_{i,j}f_{i,j}}{(u+\alpha_1+\cdots+\alpha_{i-1})^k} - \frac{A_{i,j}f_{j+1,i}}{(u+\alpha_1+\cdots+\alpha_j)^k}\right)\\ &= \sum_{i=1}^{r}\sum_{j=i}^{r}(-1)^{i+j-1}A_{i,j}\left(\frac{P\left(u+\alpha_1+\cdots+\alpha_{i-1},\alpha_i+\cdots+\alpha_j\right)}{(u+\alpha_1+\cdots+\alpha_{i-1})^k}\right.\\ &\qquad\qquad\left. - \frac{Q\left(u+\alpha_1+\cdots+\alpha_{i-1},\alpha_i+\cdots+\alpha_j\right)}{(u+\alpha_1+\cdots+\alpha_j)^k}\right).\end{aligned}$$

Note that the left-hand side of (14) is the expression in (10). To get an estimate on this expression, we use the definition of the $A_{i,j}$ and apply Lemma 3 to the right-hand side of (14). We arrive at

Lemma 5. *Let r and s be integers with $1 \le r \le s+1 \le k$. Let B and t be positive real numbers with $B \le t^{1/n}$. Suppose that $|u| \asymp t^{1/n}$ and that $|\alpha_j| \ll B$ for $1 \le j \le r$. Define $P_0, \dots, P_r$ as in (13). Then each P_i is a polynomial of degree at most $s-r+1$ in u and of total degree at most $s+((r-1)(r-2)/2)$ in the variables $u, \alpha_1, \dots, \alpha_r$. Furthermore,*

$$\frac{P_0}{u^k} + \frac{P_1}{(u+\alpha_1)^k} + \cdots + \frac{P_r}{(u+\alpha_1+\cdots+\alpha_r)^k} \ll B^{2s+1+((r-1)(r-2)/2)}t^{-(k+s+1)/n},$$

where the implied constants depend on $r, k,$ and s.

Next, we examine P_r more closely. We first obtain a lemma which will help us in this regard. Let ℓ be a non-negative integer, and consider the expression

$$E = \sum_{j=1}^{r}(-1)^j A_{r+1,j}(\alpha_j+\alpha_{j+1}+\cdots+\alpha_r)^{\ell}.$$

E is a polynomial in $\alpha_1,\dots,\alpha_r$ and can be viewed as a divided difference which is divisible by

$$\prod_{1\le\ell_1\le\ell_2\le r-1}(\alpha_{\ell_1}+\alpha_{\ell_1+1}+\cdots+\alpha_{\ell_2}).$$

It is easily checked that the remaining factor is 0 if $\ell<r-1$ and a homogeneous polynomial of degree $\ell-r+1$ if $\ell\ge r-1$. In the latter case, the next lemma asserts that whenever $\sum_{j=1}^r e_j=\ell-r+1$ with $e_1,\dots,e_r$ non-negative integers, the coefficient of $\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_r^{e_r}$ in the remaining factor is negative.

Lemma 6. *Let $r\ge 1$. Given the notation above,*

$$E=-\left(\prod_{1\le\ell_1\le\ell_2\le r-1}(\alpha_{\ell_1}+\cdots+\alpha_{\ell_2})\right)E',$$

where $E'=0$ if $\ell<r-1$ and otherwise E' is a homogeneous polynomial in $\mathbb{Z}[\alpha_1,\dots,\alpha_r]$ of degree $\ell-r+1$ having positive coefficients.

If $\ell<r-1$, the result is clear. To prove the lemma, we suppose that $\ell\ge r-1$. We use induction on r. The case $r=1$ is easily seen to be true. For $r=2$, $E=-(\alpha_1+\alpha_2)^\ell+\alpha_2^\ell$, and the result follows by applying the binomial theorem and factoring out $-\alpha_1$. For $j\in\{1,\dots,r\}$, $A_{r+1,j}$ is a polynomial which is independent of α_r. Since the product on the right-hand side above is independent of α_r, E and E' have the same degree with respect to α_r. Hence, E is of degree at most $\ell-1$ in α_r. For $r>2$, we consider the coefficient of α_r^i in E where $i\in\{0,\dots,\ell-1\}$. It is

$$\begin{aligned}&\sum_{j=1}^r(-1)^jA_{r+1,j}\binom{\ell}{i}(\alpha_j+\alpha_{j+1}+\cdots+\alpha_{r-1})^{\ell-i}\\&\quad=\binom{\ell}{i}\prod_{1\le\ell'\le r-1}(\alpha_{\ell'}+\alpha_{\ell'+1}+\cdots+\alpha_{r-1})\sum_{j=1}^{r-1}(-1)^jA'_j(\alpha_j+\alpha_{j+1}+\cdots+\alpha_{r-1})^{\ell-i-1},\end{aligned}$$

for some polynomials A'_j in $\mathbb{Z}[\alpha_1,\dots,\alpha_{r-1}]$. We clarify here that the $A_{r+1,j}$ occurring above are $A^{(r)}_{r+1,j}$. It is easily checked that $A'_j=A^{(r-1)}_{r,j}$ for $1\le j\le r-1$, and the lemma follows.

From (13), we have that

$$P_r=(-1)^r\sum_{j=1}^r(-1)^jA_{r+1,j}P(u+\alpha_1+\cdots+\alpha_r,-\alpha_j-\alpha_{j+1}-\cdots-\alpha_r).$$

We write $P(u,\alpha)=\sum_{\ell=0}^s(-1)^\ell a_\ell\alpha^\ell u^{s-\ell}$, where by Lemma 4, $a_\ell>0$ for each $\ell\in$

$\{0, 1, \dots, s\}$. Thus, from Lemma 6, we get that

$$\begin{aligned} P_r &= (-1)^r \sum_{j=1}^{r} (-1)^j A_{r+1,j} \sum_{\ell=0}^{s} a_\ell \left(\alpha_j + \alpha_{j+1} + \cdots + \alpha_r\right)^\ell \left(u + \alpha_1 + \cdots + \alpha_r\right)^{s-\ell} \\ &= (-1)^r \sum_{\ell=0}^{s} a_\ell \left(u + \alpha_1 + \cdots + \alpha_r\right)^{s-\ell} \sum_{j=1}^{r} (-1)^j A_{r+1,j} \left(\alpha_j + \alpha_{j+1} + \cdots + \alpha_r\right)^\ell \\ &= (-1)^{r+1} \left(\prod_{1 \le \ell_1 \le \ell_2 \le r-1} \left(\alpha_{\ell_1} + \cdots + \alpha_{\ell_2}\right) \right) \sum_{\ell=0}^{s} a_\ell \left(u + \alpha_1 + \cdots + \alpha_r\right)^{s-\ell} E'_\ell, \end{aligned}$$

where $E'_\ell = 0$ if $\ell < r-1$ and otherwise E'_ℓ is a homogeneous polynomial in $\mathbb{Z}[\alpha_1, \dots, \alpha_r]$ of degree $\ell - r + 1$ with positive coefficients. The next result follows.

Lemma 7. *Let r be an integer with $1 \le r \le s+1$, and let P_r be defined as in (13). Then*

$$P_r = (-1)^{r+1} \left(\prod_{1 \le \ell_1 \le \ell_2 \le r-1} \left(\alpha_{\ell_1} + \cdots + \alpha_{\ell_2}\right) \right) L(u, \alpha_1, \dots, \alpha_r),$$

where $L(u, \alpha_1, \dots, \alpha_r)$ is a homogeneous polynomial in $\mathbb{Z}[u, \alpha_1, \dots, \alpha_r]$ of degree $s - r + 1$ with positive coefficients.

Lemma 4 and Lemma 7 are considerably stronger than we require. To obtain Lemma 7, we needed that the coefficients of $P(u, \alpha)$ alternate in sign as above rather than the full strength of Lemma 4. Furthermore, we will use only that $L(0, 0, \dots, 0, \alpha_r) \ne 0$ rather than the full strenth of Lemma 7.

4. Further Preliminaries

We return now to our discussion at the end of Section 2. Fix E as in Lemma 1. Fix $J \subseteq (x, x+h]$ with $|J| = H$, where H is a real number ≥ 1 to be specified momentarily. Let $y \in J$. Recall the notation $m(u)$ in the definition of S_J. For any $u \in S_J(t, 2^n t)$ and any $\sigma \in \{\sigma_1, \dots, \sigma_n\}$, we get that

$$|(m(u) - \sigma(\mu)) - y| \le H + |\sigma(\mu)| \ll H.$$

Thus,

$$\frac{m(u) - \sigma(\mu)}{\sigma(u)^k} = \frac{y}{\sigma(u)^k} + O\left(\frac{H}{|\sigma(u)|^k}\right) = \frac{y}{\sigma(u)^k} + O\left(\frac{H}{t^{k/n}}\right). \tag{15}$$

Note that the above holds for any $u \in S_J(t, 2^n t)$. We will make use of

Lemma 8. *Let $\phi_1, \dots, \phi_d$ be any d functions from R into the real numbers having the property that for each $\alpha \in R$, there is a ϕ_j with $1 \le j \le d$ such that $|\phi_j(\alpha)| \ge 1$. Let r be a positive integer, and let $r' = r^d$. Let $\alpha_1, \dots, \alpha_{r'} \in R$. Then there exist non-negative integers $i(0), \dots, i(r)$ with $i(0) < i(1) < \cdots < i(r) \le r'$ and a $\phi \in \{\phi_1, \dots, \phi_d\}$ such that*

$$\left|\phi\left(\alpha_{i(j)+1} + \alpha_{i(j)+2} + \cdots + \alpha_{i(j+1)}\right)\right| \ge 1 \quad \text{for all } j \in \{0, \dots, r-1\}.$$

To see why Lemma 8 holds, consider $\phi_1' \in \{\phi_1, \dots, \phi_d\}$ such that $|\phi_1'(\alpha_1)| \ge 1$. Then we can take $\phi = \phi_1'$, $i(0) = 0$, and $i(1) = 1$ and find $i(2), \dots, i(r)$ as in the lemma unless there exists $i'(1) \in \{1, \dots, r' - r^{d-1}\}$ such that

$$\left|\phi_1'\left(\alpha_{i'(1)+1} + \alpha_{i'(1)+2} + \cdots + \alpha_{i'(1)+j}\right)\right| < 1 \quad \text{for all } j \in \{1, \dots, r^{d-1}\}.$$

So suppose that the latter holds. Let $\phi_2' \in \{\phi_1, \dots, \phi_d\} - \{\phi_1'\}$ with $|\phi_2'(\alpha_{i'(1)+1})| \ge 1$. We can take $\phi = \phi_2'$, $i(0) = i'(1)$, and $i(1) = i'(1) + 1$ and find $i(2), \dots, i(r)$ as in the lemma unless there exists $i'(2) \in \{i'(1) + 1, \dots, i'(1) + r^{d-1} - r^{d-2}\}$ such that

$$\left|\phi_2'\left(\alpha_{i'(2)+1} + \cdots + \alpha_{i'(2)+j}\right)\right| < 1 \quad \text{for all } j \in \{1, \dots, r^{d-2}\}.$$

We suppose the latter holds and observe that this last inequality necessarily is satisfied if we replace ϕ_2' with ϕ_1'. Continuing in this manner, we eventually obtain a positive integer $i'(d-1) \le r' - r$ such that

$$\left|\phi'\left(\alpha_{i'(d-1)+1} + \cdots + \alpha_{i'(d-1)+j}\right)\right| < 1 \quad \text{for all } j \in \{1, \dots, r\} \text{ and all } \phi' \in \{\phi_1', \dots, \phi_{d-1}'\},$$

where $\phi_1', \dots, \phi_{d-1}'$ are distinct elements of $\{\phi_1, \dots, \phi_d\}$. The one remaining element of $\{\phi_1, \dots, \phi_d\}$, say ϕ_d', must satisfy

$$\left|\phi_d'\left(\alpha_{i'(d-1)+1} + \cdots + \alpha_{i'(d-1)+j}\right)\right| \ge 1 \quad \text{for all } j \in \{1, \dots, r\}.$$

Hence, we can take $\phi = \phi_d'$, $i(0) = i'(d-1), i(1) = i'(d-1) + 1, \dots, i(r) = i'(d-1) + r$, and the lemma follows.

Divide the complex plane into disjoint quadrants $Q_1, \dots, Q_4$ defined by

$$Q_1 = \{a + bi : a = b = 0 \text{ or both } a > 0 \text{ and } b \ge 0\}, \quad Q_2 = \{a + bi : a \le 0 \text{ and } b > 0\},$$

$$Q_3 = \{a + bi : a < 0 \text{ and } b \le 0\}, \quad \text{and} \quad Q_4 = \{a + bi : a \ge 0 \text{ and } b < 0\}.$$

We will use Lemma 8 with $d = 4n$ and

$$\phi_{4i+j}(\alpha) = \begin{cases} \sigma_{i+1}(\alpha) & \text{if } \alpha \in Q_j \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \{0, \dots, n-1\}$ and $j \in \{1, 2, 3, 4\}$. Since for every $\alpha \in R$, there is a $j \in \{1, 2, 3, 4\}$ such that $\alpha \in Q_j$ and a $\sigma \in \{\sigma_1, \dots, \sigma_n\}$ such that $|\sigma(\alpha)| \ge 1$, the conditions of the lemma are met. We now fix $u' \in S_J(t, 2^n t)$. Let r and s be integers with $1 \le r \le s + 1 \le k$. We will show that for B appropriately chosen, there are $\ll 1$ numbers α' such that $||\alpha'|| \le B$ and $u' + \alpha' \in S_J(t, 2^n t)$. We may therefore assume that there are at least $r' = r^{4n} = r^d$ such non-zero α'. We denote these by $\alpha'_1, \alpha'_1 + \alpha'_2, \dots, \alpha'_1 + \cdots + \alpha'_{r'}$. Observe that $||\alpha'_1 + \cdots + \alpha'_j|| \le B$ for each $j \in \{1, \dots, r'\}$ and that $||\alpha'_i + \alpha'_{i+1} + \cdots + \alpha'_j|| \le 2B$ whenever $1 \le i \le j \le r'$. By Lemma 8, there exist $i(0), \dots, i(r)$ with $0 \le i(0) < i(1) < \cdots < i(r) \le r'$ and a $\phi \in \{\phi_1, \dots, \phi_d\}$ such that if

$$\alpha_j = \alpha'_{i(j)+1} + \alpha'_{i(j)+2} + \cdots + \alpha'_{i(j+1)} \quad \text{for all } j \in \{0, \dots, r-1\},$$

then

$$|\phi(\alpha_j)| \ge 1 \quad \text{for all } j \in \{0, \dots, r-1\}.$$

Since each α_j is necessarily in the same quadrant of the complex plane, so are any sums composed of the α_j's. In particular, we get that for some $\sigma' \in \{\sigma_1, \dots, \sigma_n\}$,

$$|\sigma'(\alpha_i + \cdots + \alpha_j)| \ge 1 \quad \text{for all } i \in \{1, \dots, r\} \text{ and } j \in \{i, \dots, r\}. \tag{16}$$

We fix α_j as above and $u = u' + \alpha'_1 + \cdots + \alpha'_{i(0)}$. Then $u + \alpha_1 + \cdots + \alpha_j \in S_J(t, 2^n t)$ for each $j \in \{1, \dots, r\}$ and $||\alpha_i + \alpha_{i+1} + \cdots + \alpha_j|| \le 2B$ for $1 \le i \le j \le r$. For every $\alpha' \in R$ with $||\alpha'|| \le B$ and $u' + \alpha' \in S_J(t, 2^n t)$, there is an α with $||\alpha|| \le 2B$ and $u + \alpha_1 + \cdots + \alpha_r + \alpha = u' + \alpha'$; thus, it suffices to show that there are $\ll 1$ values of $\alpha \in R$ such that $||\alpha|| \le 2B$ and $u + \alpha_1 + \cdots + \alpha_r + \alpha \in S_J(t, 2^n t)$. Let α_{r+1} denote any such α. Let $P_0, P_1, \dots, P_r$ be any algebraic integers in K. Then either

$$\sum_{j=0}^{r} v(u + \alpha_1 + \cdots + \alpha_j)\, P_j = 0 \tag{17}$$

or there is a $\sigma \in \{\sigma_1, \dots, \sigma_n\}$ such that

$$\left| \sigma\left(\sum_{j=0}^{r} v(u + \alpha_1 + \cdots + \alpha_j)\, P_j \right) \right| \ge 1. \tag{18}$$

Suppose for the time being that (17) does not hold, and consider any $\sigma \in \{\sigma_1, \dots, \sigma_n\}$. Then (15) implies that

(19)
$$\begin{aligned}
\sigma\left(\sum_{j=0}^{r} v(u+\alpha_1+\cdots+\alpha_j)P_j\right) &= \sum_{j=0}^{r} \frac{\sigma(E)\,(m(u+\alpha_1+\cdots+\alpha_j)-\sigma(\mu))}{\sigma(u+\alpha_1+\cdots+\alpha_j)^k}\sigma(P_j) \\
&= \sum_{j=0}^{r} \frac{\sigma(E)y}{\sigma(u+\alpha_1+\cdots+\alpha_j)^k}\sigma(P_j) + O\left(\frac{\max_{0\le j\le r}\{|\sigma(P_j)|\}H}{t^{k/n}}\right) \\
&= \sigma(E)y\,\sigma\left(\sum_{j=0}^{r} \frac{P_j}{(u+\alpha_1+\cdots+\alpha_j)^k}\right) + O\left(\frac{\max_{0\le j\le r}\{|\sigma(P_j)|\}H}{t^{k/n}}\right).
\end{aligned}$$

Observe that if $B > t^{1/n}$, then the upper bound on $|S_J(t, 2^n t)|$ in Lemma 2 will be dominated by the number 1. Now, suppose that $B \le t^{1/n}$. We consider P_j for $j \in \{0, \dots, r\}$ as in (13). Then Lemma 5 implies that

$$\max_{0 \le j \le r} \{|\sigma(P_j)|\} \ll B^{r(r-1)/2} t^{(s-r+1)/n}.$$

Since P_j is a polynomial,

$$\sigma\left(P_j(u, \alpha_1, \dots, \alpha_r)\right) = P_j\left(\sigma(u), \sigma(\alpha_1), \dots, \sigma(\alpha_r)\right),$$

and Lemma 5 implies that

$$\sigma\left(\sum_{j=0}^{r} \frac{P_j}{(u + \alpha_1 + \cdots + \alpha_j)^k}\right) \ll B^{2s+1+((r-1)(r-2)/2)} t^{-(k+s+1)/n}.$$

The idea now is to choose B and H so that (18) cannot hold and, hence, so that (17) must hold. Observe that $y \in (x, x+h]$. Let

$$w = w(r) = \frac{(r-1)(r-2)}{2}.$$

We consider

$$B \le c_6 t^{(k+s+1)/(n(2s+1+w))} x^{-1/(2s+1+w)}$$

and

$$\begin{aligned} H &\le c_7 t^{(k-s+r-1)/n} B^{-r(r-1)/2} \\ &= c_7 t^{-(2s+2-r)(r^2+2s-2k-3r+2)/(2n(2s+1+w))} x^{r(r-1)/(2(2s+1+w))}, \end{aligned}$$

where c_6 and c_7 are sufficiently small positive constants. Observe that if $r = 2$, then $w = 0$ and our choice of B and H is the same as that of Huxley and Nair given at the end of Section 2. It is easily checked that (19) implies (18) cannot hold for any $\sigma \in \{\sigma_1, \dots, \sigma_n\}$. Thus, (17) holds.

Observe that (17) holds whenever $u \in S_J(t, 2^n t)$ and $\alpha_1, \alpha_2, \dots, \alpha_r$ are such that $u + \alpha_1 + \cdots + \alpha_j \in S_J(t, 2^n t)$ and $||\alpha_j|| \le 2B$ for each $j \in \{1, \dots, r\}$. In particular, we can replace α_r by $\alpha_r + \alpha_{r+1}$ in (17), and we can replace α_{r-1} by $\alpha_{r-1} + \alpha_r$ and α_r by α_{r+1} in (17). In other words, we get the equations

$$\begin{aligned} (20) \qquad & \sum_{i=0}^{r-1} v(u + \alpha_1 + \cdots + \alpha_i)\, P_i(u, \alpha_1, \dots, \alpha_{r-1}, \alpha_r + \alpha_{r+1}) \\ & \quad + v(u + \alpha_1 + \cdots + \alpha_{r+1})\, P_r(u, \alpha_1, \dots, \alpha_{r-1}, \alpha_r + \alpha_{r+1}) = 0 \end{aligned}$$

and

$$\sum_{i=0}^{r-2} v(u+\alpha_1+\cdots+\alpha_i)\, P_i(u,\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r,\alpha_{r+1}) \\ + v(u+\alpha_1+\cdots+\alpha_r)\, P_{r-1}(u,\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r,\alpha_{r+1}) \\ + v(u+\alpha_1+\cdots+\alpha_{r+1})\, P_r(u,\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r,\alpha_{r+1}) = 0. \tag{21}$$

Recall that we are viewing $u, \alpha_1, \ldots, \alpha_r$ as being fixed and wanting to show that there are $\ll 1$ values of α_{r+1} as above. Since $u, \alpha_1, \ldots, \alpha_r$ are fixed, the values of $v(u+\alpha_1+\cdots+\alpha_i)$ for $i \in \{0, \ldots, r\}$ are fixed. We eliminate $v(u+\alpha_1+\cdots+\alpha_{r+1})$ from (20) and (21) to obtain a polynomial which any α_{r+1} as above must satisfy. More specifically, we multiply the expression on the left-hand side of (20) by $P_r(u,\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r,\alpha_{r+1})$ and subtract the product of the left-hand side of (21) with $P_r(u,\alpha_1,\ldots,\alpha_{r-1},\alpha_r+\alpha_{r+1})$. Call the result $M = M(\alpha_{r+1}) = M(u,\alpha_1,\ldots,\alpha_r,\alpha_{r+1})$. Then M is a polynomial in α_{r+1} which by Lemma 5 is of degree $\le 2s+(r-1)(r-2)$. Momentarily, we shall obtain the exact value of this degree. To show that there are $\ll 1$ choices for α_{r+1}, we observe that α_{r+1} is a root of M so that it suffices to show simply that $M \not\equiv 0$.

Lemma 4 implies that if $P = \sum_{\ell=0}^{s}(-1)^\ell a_\ell \alpha^\ell u^{s-\ell}$, then $a_s = (k-1)!/(k-s-1)! > 0$. If $i = r+1$ or $j = r$, then $A_{i,j}$ does not depend on α_r; and if $1 \le i \le r$ and $1 \le j \le r-1$, then $A_{i,j}$ is a polynomial in α_r of degree $r-2 \le s-1$. Thus, for $0 \le i \le r-1$, we get that P_i as defined in (13) is a polynomial of degree s in α_r with leading coefficient $(-1)^{r+s+i}A_{i+1,r}a_s$; and by Lemma 7, P_r is a polynomial of degree $s-r+1$ in α_r with leading coefficient $(-1)^{r+1}\left(\prod_{1\le \ell_1\le \ell_2\le r-1}(\alpha_{\ell_1}+\cdots+\alpha_{\ell_2})\right) b$ for some positive integer b. To simplify notation, we set

$$F(\alpha_1,\ldots,\alpha_{r-1}) = \prod_{1\le \ell_1\le \ell_2\le r-1}(\alpha_{\ell_1}+\cdots+\alpha_{\ell_2})$$

and

$$G(\alpha_1,\ldots,\alpha_{r-1}) = \prod_{1\le \ell\le r-1}(\alpha_{\ell}+\cdots+\alpha_{r-1}).$$

Then M is a polynomial in α_{r+1} of degree $\le 2s-r+1$, and the coefficient of α_{r+1}^{2s-r+1} in M is

$$\sum_{i=0}^{r-1}(-1)^{s+1+i}a_s b v(u+\alpha_1+\cdots+\alpha_i) \\ \times A_{i+1,r}(\alpha_1,\ldots,\alpha_{r-1},\alpha_r+\alpha_{r+1})F(\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r)$$

$$-\sum_{i=0}^{r-2}(-1)^{s+1+i}a_s b v(u+\alpha_1+\cdots+\alpha_i) \\ \times A_{i+1,r}(\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r,\alpha_{r+1})F(\alpha_1,\ldots,\alpha_{r-1})$$

$$-(-1)^{s+r}a_s b v(u+\alpha_1+\cdots+\alpha_r)A_{r,r}(\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+\alpha_r,\alpha_{r+1})F(\alpha_1,\ldots,\alpha_{r-1}).$$

To prove that $M \not\equiv 0$, it suffices to show that the expression above is not 0. We collect like terms to rewrite the expression in the form

$$(-1)^{s+1} a_s b \left(\prod_{1 \le \ell_1 \le \ell_2 \le r-2} (\alpha_{\ell_1} + \cdots + \alpha_{\ell_2}) \right) \sum_{i=0}^{r} (-1)^i D_i v(u + \alpha_1 + \cdots + \alpha_i), \tag{22}$$

and compute D_i for $0 \le i \le r$. Observe that

$$F(\alpha_1, \ldots, \alpha_{r-1}) = \left(\prod_{1 \le \ell_1 \le \ell_2 \le r-2} (\alpha_{\ell_1} + \cdots + \alpha_{\ell_2}) \right) G(\alpha_1, \ldots, \alpha_{r-1}). \tag{23}$$

In particular, this implies that the product in (23) divides both $F(\alpha_1, \ldots, \alpha_{r-1})$ and $F(\alpha_1, \ldots, \alpha_{r-2}, \alpha_{r-1} + \alpha_r)$ so that each D_i in (22) lies in $\mathbb{Z}[\alpha_1, \ldots, \alpha_r]$. Consider i fixed with $i \in \{0, \ldots, r-2\}$. We define

$$\alpha'_\ell = \begin{cases} \alpha_\ell & \text{if } 1 \le \ell \le i-1 \\ \alpha_\ell + \alpha_{\ell+1} & \text{if } \ell = i \\ \alpha_{\ell+1} & \text{if } i+1 \le \ell \le r-1, \end{cases}$$

and

$$\alpha''_\ell = \begin{cases} \alpha'_\ell & \text{if } 1 \le \ell \le r-3 \\ \alpha'_\ell + \alpha'_{\ell+1} & \text{if } \ell = r-2. \end{cases}$$

One checks that

$$\left(\prod_{1 \le \ell_1 \le \ell_2 \le r-2} (\alpha'_{\ell_1} + \cdots + \alpha'_{\ell_2}) \right) G(\alpha_1, \ldots, \alpha_{r-2}, \alpha_{r-1} + \alpha_r)$$

and

$$\left(\prod_{1 \le \ell_1 \le \ell_2 \le r-2} (\alpha''_{\ell_1} + \cdots + \alpha''_{\ell_2}) \right) G(\alpha_1, \ldots, \alpha_{r-1})$$

are each divisible by

$$\frac{1}{\alpha'_{r-1}} \left(\prod_{1 \le \ell_1 \le \ell_2 \le r-1} (\alpha'_{\ell_1} + \cdots + \alpha'_{\ell_2}) \right).$$

In addition, each has a remaining linear factor. The linear factors are $\alpha_{i+1} + \cdots + \alpha_r$ and $\alpha_{i+1} + \cdots + \alpha_{r-1}$, respectively. We easily conclude that

$$D_i = \prod_{1 \le \ell_1 \le \ell_2 \le r-1} (\alpha'_{\ell_1} + \cdots + \alpha'_{\ell_2}) \tag{24}$$

for $0 \le i \le r-2$ (where the definition of α'_ℓ depends on i). Similarly, one can check that (24) holds for $i = r-1$ and $i = r$ and α'_ℓ in each case defined as above. Thus, to show that $M \not\equiv 0$, it suffices to show that

$$\sum_{i=0}^{r} (-1)^i D_i v(u + \alpha_1 + \cdots + \alpha_i) \neq 0.$$

Analogous to (19), we get that for any $\sigma \in \{\sigma_1, \ldots, \sigma_n\}$,

$$\begin{aligned} \sigma\Bigg(\sum_{i=0}^{r}(-1)^i D_i v(u+\alpha_1+\cdots+\alpha_i)\Bigg) &= \sum_{i=0}^{r} \frac{(-1)^i \sigma(E)\left(m(u+\alpha_1+\cdots+\alpha_i) - \sigma(\mu)\right)}{\sigma(u+\alpha_1+\cdots+\alpha_i)^k}\sigma(D_i) \\ &= \sigma(E) y \sum_{i=0}^{r} \frac{(-1)^i\sigma(D_i)}{\sigma(u+\alpha_1+\cdots+\alpha_i)^k} + \sum_{i=0}^{r} O\left(\frac{|\sigma(D_i)| H}{t^{k/n}}\right). \end{aligned} \tag{25}$$

From (24), we see that the first of these last 2 sums can be viewed as a divided difference. In particular, we get that

$$\left|\sum_{i=0}^{r} \frac{(-1)^i\sigma(D_i)}{\sigma(u+\alpha_1+\cdots+\alpha_i)^k}\right| \asymp \left|\sigma\left(\prod_{1\le \ell_1\le \ell_2\le r}(\alpha_{\ell_1}+\cdots+\alpha_{\ell_2})\right)\right| t^{-(k+r)/n}. \tag{26}$$

In fact, the left-hand side of (26) can be written as the quotient of 2 polynomials in $\mathbb{Z}[u, \alpha_1, \ldots, \alpha_r]$ with the numerator divisible by the product appearing on the right-hand side of (26). Observe also that each D_i divides the product on the right-hand side of (26). Recalling (16), we take $\sigma = \sigma'$. Then

$$\left|\sigma\left(\prod_{1\le \ell_1\le \ell_2\le r}(\alpha_{\ell_1}+\cdots+\alpha_{\ell_2})\right)/D_j\right| \ge 1 \quad \text{for every } j \in \{0, \ldots, r\}.$$

Thus, we get that

$$\left|\sigma\left(\sum_{i=0}^{r} \frac{(-1)^i D_i}{(u+\alpha_1+\cdots+\alpha_i)^k}\right)\right| \gg |\sigma(D_j)| t^{-(k+r)/n} \quad \text{for every } j \in \{0, \ldots, r\}.$$

Since $y \in (x, x+h]$, we obtain that the right-hand side of (25) will be non-zero provided that

$$H \le c_8 x t^{-r/n},$$

for some sufficiently small constant c_8. Conditionally, then, we get that $M \not\equiv 0$. We apply Lemma 2 to get the following result.

Lemma 9. *Let r and s be integers with $1 \le r \le s+1 \le k$, and let $w = (r-1)(r-2)/2$. Then there exist positive constants c_6 and c_9 such that if*

$$B = c_6 t^{(k+s+1)/(n(2s+1+w))} x^{-1/(2s+1+w)}$$

and

$$H \le c_9 \min\left\{ t^{(k-s+r-1)/n} B^{-r(r-1)/2}, x t^{-r/n} \right\},$$

then whenever $J \subseteq (x, x+h]$ with $|J| \le H$,

$$|S_J(t, 2^n t)| \ll \frac{t}{B^n} + 1 \ll t^{-(k-s-w)/(2s+1+w)} x^{n/(2s+1+w)} + 1.$$

5. The Proof of the Theorem

In this section, we show how to use Lemma 9 to prove the Theorem. We consider only $n \ge 2$ since the case $n = 1$ follows from the work of Nair [10]. We take $s = k - n + r - 1$. Observe that the condition $1 \le r \le s+1 \le k$ of Lemma 9 is met provided $1 \le r \le n$ and this follows easily from the definition of r in the Theorem. We set

$$B = c_6 t^{(k+s+1)/(n(2s+1+w))} x^{-1/(2s+1+w)} \quad \text{and} \quad H = h. \tag{27}$$

We now show that the inequality on H in Lemma 9 holds provided that

$$t \le c_{10} x^{(n+r-1)/(2k-n+r)}. \tag{28}$$

With t as above, one easily checks that $h \le c_9 x t^{-r/n}$. We now show that

$$h \le c_9 t^{(k-s+r-1)/n} B^{-r(r-1)/2}. \tag{29}$$

Using the value of B given by (27), we get that the exponent on t in (29) is

$$\frac{k-s+r-1}{n} - \frac{r(r-1)(k+s+1)}{2n(2s+1+w)} = 1 - \frac{r(r-1)(2k-n+r)}{n(4k-4n+r^2+r)}$$

$$= \frac{(2n - r(r-1))(2k-2n+r)}{n(4k-4n+r^2+r)} > 0.$$

Thus, the right-hand side of (29) obtains its minimum when t is minimal. Recalling from Section 2 that we are interested in $t \ge c_2 T \gg h\sqrt{\log x}$, we use the definition of h and (27) to obtain that the right-hand side of (29) is $\gg c^{-\ell_1} h (\sqrt{\log x})^{\ell_2}$ where $\ell_1 = -r(r-1)(k+s+1)/(2n(2s+1+w))$ and where ℓ_2 is the exponent on t in (29) given above and, hence, positive. Since x is sufficiently large, (29) holds.

We get from Lemma 9 that

$$|S(t, 2^n t)| \ll t^{-(k-s-w)/(2s+1+w)} x^{n/(2s+1+w)} + 1 \quad \text{if (28) holds.}$$

Observe that if $u^k v = E(m - \mu)$ as in Lemma 1, then $|u| \ll x^{1/k}$. Thus, $|S(t, 2^n t)| = 0$ unless $t \ll x^{n/k}$. We suppose now that $t \ll x^{n/k}$. It is easily checked then that the first term on the right-hand side above is > 1. Also, $k - s - w > 0$. Hence,

$$\left| S(c_2^n T, c_{10} x^{(n+r-1)/(2k-n+r)}) \right| \ll T^{-(k-s-w)/(2s+1+w)} x^{n/(2s+1+w)} \ll x^{n/(2k-n+r)}.$$

Although our main application of Lemma 9 is that given above, we consider applying the lemma with a different value of r, namely $r = 1$. It is necessary, however, to avoid altering the value of h which depends on the value of r given above. In other words, we consider $h = cx^{n/(2k-n+r)}$ with r fixed as in the statement of the Theorem, and we consider Lemma 9 with the role of r replaced with 1 and with $s = k - n$. One easily gets that the lemma applies and

$$\left| S(c_{10} x^{(n+r-1)/(2k-n+r)}, \infty) \right|$$

$$\ll \left(x^{(n+r-1)/(2k-n+r)} \right)^{-n/(2k-2n+1)} x^{n/(2k-2n+1)} \ll x^{n/(2k-n+r)}.$$

The Theorem now follows.

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