THE DISTRIBUTION OF FRACTIONAL PARTS WITH APPLICATIONS TO GAP RESULTS IN NUMBER THEORY

Michael Filaseta*and Ognian Trifonov**

1. Introduction

Let δ and N be positive real numbers, and let $f : \mathbb{R} \mapsto \mathbb{R}$ be any function. In this paper, we will obtain estimates for the size of the set $\{u \in (N, 2N] : ||f(u)|| < \delta\}$, where u represents an integer and $||f(u)||$ represents the distance from $f(u)$ to the nearest integer. Thus, the set consists of $u \in (N, 2N]$ for which $\{f(u)\}$, the fractional part of $f(u)$, lies in $I = [0, \delta) \cup (1 - \delta, 1)$. The function f may depend on other parameters than u and, in particular, N. How good an estimate one can make depends on how well-behaved the function f is as well as the size of δ. We will obtain two different theorems estimating the size of this set. The first of these two theorems will be of a fairly general nature and is motivated by the authors' work in [12] and by work of Huxley [22] and of Huxley and Sargos [24]; the second of these two theorems will be less general and is motivated by the authors' joint work in [11] and the second author's work in [49].

We will show how these two theorems can be used to obtain gap results for $k-$free numbers (integers n such that if p is a prime, then p^k does not divide n), for squarefull numbers (integers n such that if p is a prime dividing n, then p^2 divides n), and for the number of non-isomorphic abelian groups of a given order. For the remainder of this introduction we elaborate on these gap results.

Theorem 1. *Let k be an integer ≥ 2. Then there is a constant $c = c(k)$ such that if $h = cx^{1/(2k+1)} \log x$ and x is sufficiently large, then the interval $(x, x+h]$ contains a $k-$free number. Furthermore, if $h = x^{1/(2k+1)}(\log x)g(x)$ where $g(x)$ is any function increasing to infinity with x, then the number of $k-$free numbers in $(x, x + h]$ is $h/\zeta(k) + o(h)$.*

Theorem 1 for $k = 2$ was first established by the authors in [11] and for general k was first established by the second author in [49]. For $k = 2$, this result improves on work by Fogels [13], Roth [39], Richert [38], Rankin [37], Schmidt [40], Graham and Kolesnik [17], the second author [46,47], the first author [6], and the authors jointly [10]. For general k, this result improves on work by Halberstam and Roth [18], the first author [5], and the second author [48].

*Research was supported by NSA Grant MDA904-92-H-3011 and by NSF Grant DMS-9400937.

**Research supported by NSF EPSCoR Grant EHR 9108772, by Grant 403/94 from the Bulgarian Academy of Sciences, and by the EC Human Capital and Mobility Programme.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Theorem 2. *Let $a(n)$ denote the number of non-isomorphic abelian groups of order n. For k a positive integer, let $A_k = \{n \in \mathbb{Z}^+ : a(n) = k\}$. Let $h = x^{1/5}(\log x)g(x)$ where $g(x)$ is any function increasing to infinity with x. Then there is a constant P_k such that the interval $(x, x+h]$ contains $P_k h + o(h)$ elements of A_k.*

Theorem 2 improves on work of Ivić [26] and Krätzel [29,30]. The theorem is essentially due to Li Hongze [31] who showed that one can take $h = x^{(1/5)+\epsilon}$ above for any $\epsilon > 0$. The approach of Li is based on using the exponent pair $(1/6, 2/3)$ and estimates from work on gaps between squarefree and cubefree numbers from [11] and [18]. Our approach here, and elsewhere in the paper, is based solely on elementary techniques. We also show that a result of the type given in Theorem 1 for $k = 2$ is equivalent to an analogous result (with the same h) for Theorem 2. To show this equivalence, we make use of a result of Schinzel (cf. [4],[43]) that the unrestricted partition function $p(n)$ has infinitely many distinct prime divisors as n varies over the positive integers.

Theorem 3. *Let $Q(x)$ denote the number of squarefull numbers $\le x$. Then for*

$$\frac{5}{39} = 0.1282\cdots < \theta < \frac{1}{2},$$

we have that

$$Q(x + x^{(1/2)+\theta}) - Q(x) = \frac{\zeta(3/2)}{2\zeta(3)} x^{\theta} + o(x^{\theta}).$$

This is the main result of the authors in [12]. A classical result of Bateman and Grosswald [2] would imply the same result with $\theta > 1/6$. Theorem 3 is an improvement on work by Shiu [44,45], Schmidt [41,42], Jia [27,28], Liu [32,33], and Heath-Brown [19].

Theorem 4. *Let $s_1, s_2, \ldots$ denote the squarefree numbers in ascending order. Then for*

$$0 \le \gamma < \frac{43}{13} = 3.30769\ldots,$$

there is a constant $B(\gamma)$ such that

$$\sum_{s_{n+1} \le x} (s_{n+1} - s_n)^{\gamma} = B(\gamma)x + o(x).$$

Theorem 5. *Let k be an integer ≥ 3. Let $s_1, s_2, \ldots$ denote the $k-$free numbers in ascending order. For $0 \le \gamma < 2k - 1$, there is a constant $B(\gamma, k)$ such that*

$$\sum_{s_{n+1} \le x} (s_{n+1} - s_n)^{\gamma} = B(\gamma, k)x + o(x).$$

Theorem 4 improves on work by Erdős [3], Hooley [21], and the first author [7]. Theorem 5 is the $k-$free version of Theorem 4 and improves on work of Hooley [21] and Graham [15]. Theorem 5 is a slight improvement over Graham who had the result for $0 \le \gamma <$

$2k-2+(4/(k+1))$. The same result for $\gamma > 2k+1$ would imply an improvement on Theorem 1, and it is of some interest to determine how close to $2k+1$ one can take γ.

In this introduction, we have indicated some of the extent of the approaches in this paper. The main results concerning the distribution of fractional parts and the methods used are discussed in detail in the next two sections. It is worth noting here that these same ideas can be extended to algebraic number fields to obtain results about $k-$free values of polynomials and binary forms. Such polynomial results have been obtained by Nair [35,36], Huxley and Nair [23], and the first author [8]. In particular, Nair [35] showed that if k is an integer $\geq (\sqrt{2}-(1/2))g$, then any irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree g with $\gcd(f(m) : m \in \mathbb{Z})$ $k-$free is such that $f(m)$ is $k-$free for infinitely many integers m. The first author [9] has obtained the analogous result for irreducible binary forms $f(x,y) \in \mathbb{Z}[x,y]$ and $k \geq (1/2)(\sqrt{2}-(1/2))g$.

2. Divided Differences and a Result on Fractional Parts

Before continuing, we mention briefly some notation that will be used throughout the paper.

$||x|| = \min\{|x-m| : m \text{ is an integer}\}$.

$[x[$ denotes the nearest integer to x.

$f(u) \ll g(u)$, $g(u) \gg f(u)$, or $f(u) = O(g(u))$ will mean that there is a constant $c' > 0$ such that $|f(u)| \leq c'|g(u)|$ whenever u is sufficiently large. A subscript will be used to indicate the dependence of the constant on some fixed parameters; for example $f(u) \ll_\epsilon g(u)$ or $f(u) = O_\epsilon(g(u))$ indicates that c' may depend on ϵ.

$f(u) = o(g(u))$ will mean that $\lim_{u\to\infty}(f(u)/g(u)) = 0$.

$f(u) \sim g(u)$ will mean that $\lim_{u\to\infty}(f(u)/g(u)) = 1$.

$f(u) \asymp g(u)$ will mean that there are constants $c_1' > 0$ and $c_2' > 0$ such that $c_1'|g(u)| \leq |f(u)| \leq c_2'|g(u)|$.

In this section and the next, we develop the two main results of this paper from which the theorems of the previous section will follow either directly or indirectly. We also indicate how exponential sum estimates can be used to obtain similar results. An interesting aspect of the results mentioned in the previous section, however, is that although exponential sum estimates have been used in the past to obtain similar theorems, we will make no use of exponential sum estimates in this paper. Nevertheless, it is of some interest to understand the role of exponential sums for such results.

For all of the results mentioned in the previous section, we will have $f(u) = X/u^s$ where X is a new parameter and where s is a positive rational number. The value of δ will in each case be "small," and this will be crucial for the use of the more general theorems discussed in this section and the next. In particular, the larger δ is, one can in general expect to be able to make less use of the theorems in this paper and better use of the methods of exponential sums.

Exponential sums serve immediately to make estimates of the type we want because of the following:

Let $\delta \in (0,1/2)$. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be any function. Let S be a set of positive integers.

Then for any positive integer $J \le 1/(4\delta)$, we get that

$$|\{u \in S : ||f(u)|| < \delta\}| \le \frac{\pi^2}{2(J+1)} \sum_{1 \le j \le J} \left| \sum_{u \in S} e\left(jf(u)\right) \right| + \frac{\pi^2}{4(J+1)}|S|.$$

A particularly simple proof of this result can be found in [6]. If one also makes use of Abel summation, one can replace the above estimate with

$$|\{u \in S : ||f(u)|| < \delta\}| \le \frac{\pi^2}{2(J+1)} \max_{1 \le k \le J} \left\{ \left| \sum_{u \in S} \sum_{j=1}^{k} e\left(jf(u)\right) \right| \right\} + \frac{\pi^2}{4(J+1)}|S|.$$

Both of these enable one to apply exponential sum techniques. In particular, one can apply the theory of exponent pairs in the first case or multi-dimensional estimates in the second case. An excellent reference on obtaining such estimates is a recent book by Graham and Kolesnik [16]. In addition, for monomials f such as we will be considering, one can make use of recent work by Fouvry and Iwaniec [14].

Our interest here, again, is not in the use of exponential sums. We will make use of finite differences. Finite differences arise in the theory of exponential sums. It is not difficult, for example, to transform a sum over u of $e(f(u))$ to a sum over u and a of $e(f(u+a) - f(u))$. The expression $f(u+a) - f(u)$ is a difference, and we can view it as being close to $af'(u)$ if f is well-behaved and from this obtain estimates for exponential sums. We can furthermore arrive at higher order differences to estimate such sums. The advantage of avoiding the use of exponential sums is that in so doing one seemingly can make more use of certain arithmetical considerations in order to obtain better estimates of the type needed. Equally important is that one need not restrict oneself to the type of differences that arise in exponential sum estimates. A crucial aspect to the proof of our next result, for example, is an application of divided differences which we discuss next. A theory of exponential sum estimates based on the use of such differences would be of interest, but none currently exists.

If f is a continuous function defined on the reals and $x_0, x_1, \dots, x_n$ are $n+1$ distinct real numbers, then there is a unique polynomial $g(x)$ of degree $\le n$ such that $g(x_j) = f(x_j)$ for each $j \in \{0, 1, \dots, n\}$. The coefficient of x^n in $g(x)$ is defined to be an nth order divided difference for f, and we denote it by $f[x_0, x_1, \dots, x_n]$. Divided differences are classical in the theory of numerical methods. Below we give the necessary background for our use of them.

Let s be a positive integer. For arbitrary distinct real numbers $\beta_0, \beta_1, \dots, \beta_s$, we set

$$G^{(s)} = G^{(s)}(\beta_0, \beta_1, \dots, \beta_s) = \prod_{0 \le i < j \le s} (\beta_j - \beta_i).$$

For j in $\{0, 1, \dots, s\}$, we define

$$G_j^{(s)} = G_j^{(s)}(\beta_0, \beta_1, \dots, \beta_s) = G^{(s-1)}(\beta_0, \beta_1, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_{s-1}, \beta_s),$$

$$M_j^{(s)} = M_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s) = (-1)^{s-j} G_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s),$$

and

$$D_j^{(s)} = D_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s) = \prod_{\substack{0 \le i \le s \\ i \ne j}} (\beta_j - \beta_i) = \frac{G^{(s)}(\beta_0, \beta_1, \ldots, \beta_s)}{M_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s)}.$$

To clarify our use of subscripts, we note that in the definition $G_j^{(s)}$ we are discarding the $(j+1)$st argument so that for example

$$G_j^{(s)}(\beta_1, \beta_2, \ldots, \beta_{s+1}) = \prod_{\substack{0 \le i < \ell \le s \\ i \ne j+1, \ell \ne j+1}} (\beta_\ell - \beta_i).$$

For our purposes, we fix $\beta_0 = 0$. We now formulate four lemmas. The first two are well-known and can be found in [25].

Lemma 1. *With the notation above,*

$$f[u, u+\beta_1, \ldots, u+\beta_s] = \sum_{j=0}^{s} \frac{f(u+\beta_j)}{D_j^{(s)}} = \sum_{j=0}^{s} \frac{f(u+\beta_j) M_j^{(s)}}{G^{(s)}}$$

and

$$f[u, u+\beta_1, \ldots, u+\beta_s] = \frac{f[u+\beta_1, u+\beta_2, \ldots, u+\beta_s] - f[u, u+\beta_1, \ldots, u+\beta_{s-1}]}{\beta_s}.$$

Lemma 2. *Suppose that $0 < \beta_1 < \beta_2 < \cdots < \beta_s$ and that f has a continuous derivative of order s in $[u, u+\beta_s]$. Then there is a number $\xi \in (u, u+\beta_s)$ for which*

$$f[u, u+\beta_1, \ldots, u+\beta_s] = \frac{f^{(s)}(\xi)}{s!}.$$

Lemma 3. *Let $s \ge 2$, and let $\beta_1, \beta_2, \ldots, \beta_s$ be distinct positive integers. Let*

$$d_1 = \beta_1 \beta_2 \cdots \beta_{s-1}, \quad d_2 = (\beta_s - \beta_1)(\beta_s - \beta_2) \cdots (\beta_s - \beta_{s-1}), \quad \text{and} \quad d = \gcd(d_1, d_2).$$

Then for every $j \in \{0, 1, \ldots, s\}$, d is a factor of $M_j^{(s)}$, and for every $j \in \{1, 2, \ldots, s-1\}$,

$$M_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s) = d_1 M_{j-1}^{(s-1)}(\beta_1, \beta_2, \ldots, \beta_s) - d_2 M_j^{(s-1)}(\beta_0, \beta_1, \ldots, \beta_{s-1}).$$

Proof. Clearly, $G_s^{(s)} = \prod_{0 \le i < j \le s-1} (\beta_j - \beta_i)$ is divisible by d_1 and $G_0^{(s)} = \prod_{1 \le i < j \le s} (\beta_j - \beta_i)$ is divisible by d_2. Hence, d divides $M_j^{(s)}$ for $j = 0$ and $j = s$. For $j \in \{1, 2, \ldots, s-1\}$, one checks directly that

$$d_1 G_{j-1}^{(s-1)}(\beta_1, \beta_2, \ldots, \beta_s) = \frac{\beta_j}{\beta_s} G_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s)$$

and

$$d_2 G_j^{(s-1)}(\beta_0, \beta_1, \ldots, \beta_{s-1}) = \frac{\beta_s - \beta_j}{\beta_s} G_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s).$$

Hence,

$$G_j^{(s)}(\beta_0, \beta_1, \ldots, \beta_s) = d_1 G_{j-1}^{(s-1)}(\beta_1, \beta_2, \ldots, \beta_s) + d_2 G_j^{(s-1)}(\beta_0, \beta_1, \ldots, \beta_{s-1}).$$

The result follows. ■

Lemma 4. *If n is an integer and $n = \theta + \epsilon$, then one of the following must hold:*

(i) $|\theta| \geq 1/2$.

(ii) $|\theta| \leq |\epsilon|$.

If $|\epsilon| < 1/2$, then n is non-zero in case (i) and $n = 0$ in case (ii).

The argument for Lemma 4 is clear. We have stated it here simply to accentuate its importance as this simple lemma will play a major role in our arguments.

We state the main result of this section next. In the statement of the result, we will make use of two numbers r and k. Throughout the remainder of this section, there will be various implied constants (such as in the asymptotic formulas $f^{(j)}(u) \asymp TN^{-j}$ in the statement of our next result); all such constants may depend on r but are independent of k.

Theorem 6. *Let $N > 1$. Let r be an integer ≥ 3. Let T be a positive real number. Suppose that f is a function with at least r derivatives and with $f^{(j)}(u) \asymp TN^{-j}$ for $j \in \{r-2, r-1, r\}$ and $u \in (N, 2N]$. Let δ be a positive real number with*

$$\delta < k \min\{TN^{-r+2}, T^{(r-4)/(r-2)}N^{-r+3} + TN^{-r+1}\},$$

for some sufficiently small constant k depending on r and the implied constants in the asymptotic formulas $f^{(j)}(u) \asymp TN^{-j}$ above. Let

$$S = \{u \in (N, 2N] \cap \mathbb{Z} : ||f(u)|| < \delta\}.$$

Then

$$|S| \ll T^{2/(r(r+1))}N^{(r-1)/(r+1)} + N\delta^{2/((r-1)(r-2))} + N\left(\delta TN^{1-r}\right)^{1/(r^2-3r+4)}.$$

To prove Theorem 6, we first consider several key steps which we designate as lemmas. For each lemma, we will assume that the conditions in the theorem hold.

Lemma 5. *For some positive constant c_1 depending only on r and the implied constants in $f^{(r-2)}(u) \asymp TN^{-r+2}$, there exists a set S' satisfying*

(i) $S' \subseteq S$,

(ii) $|S| \leq (r-1)(|S'|+1)$,

and

(iii) any two distinct elements of S' differ by at least $c_1 N^{2/(r-1)}T^{-2/((r-1)(r-2))}$.

Proof. Let $u, u+\beta_1, \ldots, u+\beta_{r-2}$ denote any $r-1$ consecutive elements of S so that $0 < \beta_1 < \beta_2 < \cdots < \beta_{r-2}$. We will establish that

$$\beta_{r-2} > c_1 N^{2/(r-1)}T^{-2/((r-1)(r-2))}$$

for some c_1 as above from which the lemma will follow by choosing the elements of S' to be every $(r-1)$st element of S.

From Lemma 1 and Lemma 2, we deduce that

$$f[u, u+\beta_1, \ldots, u+\beta_{r-2}] = \sum_{j=0}^{r-2} \frac{f(u+\beta_j)M_j^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2})}{G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2})} = \frac{f^{(r-2)}(\xi)}{(r-2)!}$$

for some $\xi \in (u, u+\beta_{r-2})$. Therefore,

$$(1) \qquad \sum_{j=0}^{r-2} f(u+\beta_j) M_j^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) = \frac{G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) f^{(r-2)}(\xi)}{(r-2)!}.$$

Since $u, u+\beta_1, \ldots, u+\beta_{r-2}$ are in S,

$$(2) \qquad ||f(u+\beta_j)|| < \delta \quad \text{for } 0 \leq j \leq r-2.$$

The definitions of $G^{(s)}$ and $M_j^{(s)}$ easily imply

$$\left| M_j^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) \right| \leq G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) \quad \text{for } 0 \leq j \leq r-2.$$

Also, by the conditions in the theorem, $\delta < kTN^{-r+2} \ll kf^{(r-2)}(\xi)$. Hence, the left-hand side of (1) differs from an integer by

$$< \delta \sum_{j=0}^{r-2} \left| M_j^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) \right| \leq \delta(r-1) G^{(r-2)} \ll kG^{(r-2)} f^{(r-2)}(\xi).$$

From Lemma 4 and (1), we deduce that for k sufficiently small,

$$\frac{G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) f^{(r-2)}(\xi)}{(r-2)!} \geq \frac{1}{2}.$$

The lemma follows upon observing that $G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) \leq \beta_{r-2}^{(r-1)(r-2)/2}$ and that $f^{(r-2)}(\xi) \ll TN^{-r+2}$. ■

We observe before continuing that to prove Lemma 5, we used only the conditions $f^{(r-2)} \asymp TN^{-r+2}$ and $\delta < kTN^{-r+2}$ of Theorem 6.

Fix positive integers $a_1, a_2, \ldots, a_{r-1}$, and suppose that

$$(3) \qquad a_j \leq A \leq c_2 \delta^{-2/((r-1)(r-2))},$$

where $c_2 > 0$ is a sufficiently small constant depending only on r and the implied constants in the conditions $f^{(j)} \asymp TN^{-j}$ of the theorem. Set $\beta_0 = 0$ as before, and set

$$\beta_j = a_1 + a_2 + \cdots + a_j \text{ for } j \in \{1, 2, \ldots, r-1\}.$$

Let $T(a_1, a_2, \ldots, a_{r-1})$ denote the set of integers u such that the numbers $u, u+\beta_1, u+\beta_2, \ldots, u+\beta_{r-1}$ are consecutive elements of S'. Set

$$t(a_1, a_2, \ldots, a_{r-1}) = |T(a_1, a_2, \ldots, a_{r-1})|.$$

Let u and $u+c$ denote two consecutive elements of $T(a_1, a_2, \ldots, a_{r-1})$. Define $g(x) = f(x+c) - f(x)$. Let d_1, d_2, and d be as defined in Lemma 3 with $s = r-1$.

Lemma 6. *There are real numbers $\epsilon, \epsilon_1, \epsilon_2, \eta_1$, and η_2 with absolute value $< 1/4$ and integers m, m_1, m_2, ℓ_1, and ℓ_2 such that*

$$G^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1}) f[u, u+\beta_1, \ldots, u+\beta_{r-1}] = m_1 + \epsilon_1, \tag{4}$$

$$G^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1}) f[u+c, u+c+\beta_1, \ldots, u+c+\beta_{r-1}] = m_2 + \epsilon_2, \tag{5}$$

$$G^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1}) g[u, u+\beta_1, \ldots, u+\beta_{r-1}] = m + \epsilon, \tag{6}$$

$$G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2}) g[u, u+\beta_1, \ldots, u+\beta_{r-2}] = \ell_1 + \eta_1, \tag{7}$$

and

$$G^{(r-2)}(\beta_1, \beta_2, \ldots, \beta_{r-1}) g[u+\beta_1, u+\beta_2, \ldots, u+\beta_{r-1}] = \ell_2 + \eta_2. \tag{8}$$

Furthermore,

$$m = m_2 - m_1 = d_1\ell_2 - d_2\ell_1, \quad \epsilon = \epsilon_2 - \epsilon_1 = d_1\eta_2 - d_2\eta_1, \quad d|m_1, \quad d|m_2, \quad \text{and} \quad d|m.$$

Proof. Since $u+\beta_j$ and $u+c+\beta_j$ are in S' for $j \in \{0, 1, \ldots, r-1\}$, we deduce for each such j that

$$||f(u+\beta_j)|| < \delta, \quad ||f(u+c+\beta_j)|| < \delta, \quad \text{and } ||g(u+\beta_j)|| < 2\delta. \tag{9}$$

From Lemma 1, we obtain that the left-hand side of (4) differs from an integer, which we call m_1, by

$$< \delta \sum_{j=0}^{r-1} |M_j^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1})| \ll \delta \beta_{r-1}^{(r-1)(r-2)/2} \ll \delta A^{(r-1)(r-2)/2},$$

where the implied constant depends on r but is independent of c_2. Thus, with c_2 sufficiently small, we obtain from (3) that (4) holds with $|\epsilon_1| < 1/8$. Also, recalling $[x[$ denotes the nearest integer to x, we deduce

$$\epsilon_1 = \sum_{j=0}^{r-1} \left(f(u+\beta_j) - [f(u+\beta_j)[\right) \, M_j^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1})$$

and

$$m_1 = \sum_{j=0}^{r-1} [f(u+\beta_j)[\, M_j^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1}).$$

We obtain from Lemma 3 that $d|m_1$. In a similar manner, we obtain (5) with $|\epsilon_2| < 1/8$ and with m_2 an integer divisible by d.

From Lemma 1, we deduce

$g[u, u+\beta_1, \dots, u+\beta_{r-1}] = f[u+c, u+c+\beta_1, \dots, u+c+\beta_{r-1}] - f[u, u+\beta_1, \dots, u+\beta_{r-1}]$.

Hence, (6) follows from (4) and (5) with

$$m = m_2 - m_1, \quad \epsilon = \epsilon_2 - \epsilon_1, \quad \text{and} \quad |\epsilon| < 1/4.$$

Since $d|m_1$ and $d|m_2$, we deduce $d|m$. Observe that an argument similar to that given in the previous paragraph implies from (9) that

$$\sum_{j=0}^{r-1} \Big(||g(u+\beta_j)|| \times |M_j^{(r-1)}(\beta_0, \beta_1, \dots, \beta_{r-1})| \Big) < 1/4.$$

We obtain here that

$$m = \sum_{j=0}^{r-1} [g(u+\beta_j)[\, M_j^{(r-1)}(\beta_0, \beta_1, \dots, \beta_{r-1})$$

and

$$\epsilon = \sum_{j=0}^{r-1} (g(u+\beta_j) - [g(u+\beta_j)[) \; M_j^{(r-1)}(\beta_0, \beta_1, \dots, \beta_{r-1}).$$

We will use this information momentarily.

One can establish (7) and (8) for some real numbers η_1 and η_2 having absolute value $< 1/4$ and some integers ℓ_1 and ℓ_2 by applying arguments very similar to those used to obtain (4) and (5). We omit the details. Furthermore, analogous to our expressions for m and ϵ above, we obtain

$$\ell_1 = \sum_{j=0}^{r-2} [g(u+\beta_j)[\, M_j^{(r-2)}(\beta_0, \beta_1, \dots, \beta_{r-2}),$$

$$\ell_2 = \sum_{j=0}^{r-2} [g(u+\beta_{j+1})[\, M_j^{(r-2)}(\beta_1, \beta_2, \dots, \beta_{r-1}),$$

$$\eta_1 = \sum_{j=0}^{r-2} (g(u+\beta_j) - [g(u+\beta_j)[) \; M_j^{(r-2)}(\beta_0, \beta_1, \dots, \beta_{r-2}),$$

and

$$\eta_2 = \sum_{j=0}^{r-2} (g(u+\beta_{j+1}) - [g(u+\beta_{j+1})[) \; M_j^{(r-2)}(\beta_1, \beta_2, \dots, \beta_{r-1}).$$

Observe that

$$M_{r-1}^{(r-1)}(\beta_0, \beta_1, \dots, \beta_{r-1}) = d_1 M_{r-2}^{(r-2)}(\beta_1, \beta_2, \dots, \beta_{r-1})$$

and

$$M_0^{(r-1)}(\beta_0, \beta_1, \dots, \beta_{r-1}) = -d_2 M_0^{(r-2)}(\beta_0, \beta_1, \dots, \beta_{r-2}).$$

From the last part of Lemma 3, we immediately deduce that $m = d_1\ell_2 - d_2\ell_1$ and $\epsilon = d_1\eta_2 - d_2\eta_1$. This completes the proof. ■

Lemma 7. *Given the notation of Lemma 6, none of* m_1, m_2, ℓ_1, *and* ℓ_2 *is equal to zero. Also,* $d \ll G^{(r-1)}(\beta_0, \ldots, \beta_{r-1})\dfrac{T}{N^{r-1}}$.

Proof. We use that

$$\delta < k\left(\frac{T^{(r-4)/(r-2)}}{N^{r-3}} + \frac{T}{N^{r-1}}\right) \le 2k \max\left\{\frac{T^{(r-4)/(r-2)}}{N^{r-3}}, \frac{T}{N^{r-1}}\right\},$$

and divide the argument into two cases.

Case I. $\delta < 2kT/N^{r-1}$.

Lemma 2 and the conditions in the theorem imply that

$$f[u, u+\beta_1, \ldots, u+\beta_{r-1}] \asymp \frac{T}{N^{r-1}}.$$

From the proof of Lemma 6, we deduce

$$|\epsilon_1| \le \delta \sum_{j=0}^{r-1} |M_j^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1})| \le \delta r G^{(r-1)}(\beta_0, \ldots, \beta_{r-1})$$

$$< 2krG^{(r-1)}(\beta_0, \ldots, \beta_{r-1})\frac{T}{N^{r-1}}$$

$$\ll kG^{(r-1)}(\beta_0, \ldots, \beta_{r-1}) f[u, u+\beta_1, \ldots, u+\beta_{r-1}].$$

Next, we apply Lemma 4 using $n = m_1$ and (4). Since k is sufficiently small, the above inequality on $|\epsilon_1|$ implies that (ii) cannot hold; hence, (i) holds. Since $|\epsilon_1| < 1/4$, we obtain that $m_1 \neq 0$. Observe also that

$$m_1 \asymp G^{(r-1)}(\beta_0, \ldots, \beta_{r-1})\frac{T}{N^{r-1}}. \tag{10}$$

A similar argument can be used to deduce that $m_2 \neq 0$. Lemma 1 and Lemma 2 (together with the Mean Value Theorem) imply

$$g[u, u+\beta_1, \ldots, u+\beta_{r-2}]$$

$$= f[u+c, u+c+\beta_1, \ldots, u+c+\beta_{r-2}] - f[u, u+\beta_1, \ldots, u+\beta_{r-2}]$$

$$\asymp c\frac{T}{N^{r-1}} \gg \frac{T}{N^{r-1}}.$$

It follows as above that $\ell_1 \neq 0$, and a similar argument gives $\ell_2 \neq 0$. The inequality on d follows from (10) and $d|m_1$.

Case II. $\delta < 2kT^{(r-4)/(r-2)}/N^{r-3}$.

In this case we use that

$$|M_j^{(r-1)}(\beta_0,\dots,\beta_{r-1})| = \frac{G^{(r-1)}(\beta_0,\dots,\beta_{r-1})}{|D_j^{(r-1)}(\beta_0,\dots,\beta_{r-1})|}.$$

Since $u,\dots,u+\beta_{r-1}$ are elements of S', Lemma 5 implies that the distance between $u+\beta_i$ and $u+\beta_j$ is $\gg N^{2/(r-1)}T^{-2/((r-1)(r-2))}$ for $i \neq j$. Therefore,

$$|D_j^{(r-1)}| \gg N^2T^{-2/(r-2)}.$$

Thus, in this case, we obtain

$$|\epsilon_1| \le \delta \sum_{j=0}^{r-1} |M_j^{(r-1)}(\beta_0,\beta_1,\dots,\beta_{r-1})| \ll \delta G^{(r-1)}(\beta_0,\dots,\beta_{r-1})N^{-2}T^{2/(r-2)}$$

$$\ll 2kG^{(r-1)}(\beta_0,\dots,\beta_{r-1})\frac{T}{N^{r-1}} \ll kG^{(r-1)}(\beta_0,\dots,\beta_{r-1})f[u,u+\beta_1,\dots,u+\beta_{r-1}].$$

As in Case I, we deduce that $m_1 \neq 0$ and that (10) holds. Again, $m_2 \neq 0$ follows in a similar manner, and one obtains the upper bound on d. The estimate

$$c \ge \beta_1 \gg N^{2/(r-1)}T^{-2/((r-1)(r-2))}$$

can be used in this case to establish $\ell_1 \neq 0$ and $\ell_2 \neq 0$. ■

Proof of Theorem 6. Let

$$\theta = G^{(r-1)}(\beta_0,\beta_1,\dots,\beta_{r-1})g[u,u+\beta_1,\dots,u+\beta_{r-1}].$$

Using (6), we deduce that either (i) or (ii) of Lemma 4 holds.

Suppose (i) holds. By Lemma 6, $|\epsilon| \le 1/4$. By Lemma 4, $m \neq 0$. Since $d|m$, we obtain $d \le |m|$. Hence,

$$G^{(r-1)}(\beta_0,\dots,\beta_{r-1})|g[u,u+\beta_1,\dots,u+\beta_{r-1}]| \ge \frac{d}{2}.$$

From Lemma 1, Lemma 2, and the Mean Value Theorem,

$$g[u,u+\beta_1,\dots,u+\beta_{r-1}]$$

$$= f[u+c,u+c+\beta_1,\dots,u+c+\beta_{r-1}] - f[u,u+\beta_1,\dots,u+\beta_{r-1}] \asymp \frac{cT}{N^r}.$$

Therefore,

$$c \gg B_1 := \frac{dN^r}{TG^{(r-1)}(\beta_0,\dots,\beta_{r-1})}.$$

From Lemma 7, $B_1 \ll N$.

Now, suppose (ii) holds. Here Lemma 4 implies $m = 0$ so that $|\theta| = |\epsilon|$. Thus,

$$|\epsilon| = G^{(r-1)}(\beta_0, \ldots, \beta_{r-1})|g[u, u+\beta_1, \ldots, u+\beta_{r-1}]| \asymp G^{(r-1)}(\beta_0, \ldots, \beta_{r-1})\frac{cT}{N^r}.$$

From the proof of Lemma 6,

$$|\epsilon| \ll \delta \sum_{j=0}^{r-1} |M_j^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1})|.$$

Hence,

$$c \ll B_2 := \frac{\delta \sum_{j=0}^{r-1} |M_j^{(r-1)}(\beta_0, \beta_1, \ldots, \beta_{r-1})|N^r}{TG^{(r-1)}(\beta_0, \ldots, \beta_{r-1})}.$$

Since $m = 0$, Lemma 6 implies $d_1\ell_2 = d_2\ell_1$ so that $(d_1/d)|\ell_1$. By Lemma 7, $\ell_1 \neq 0$. Hence,

$$|\ell_1| \geq d_1/d. \tag{11}$$

Now, we use (7) and Lemma 4. Since $|\eta_1| < 1/4$ and $\ell_1 \neq 0$, we obtain from (11) that

$$G^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2})\,|g[u, u+\beta_1, \ldots, u+\beta_{r-2}]| \geq \frac{d_1}{2d}.$$

Since $g[u, u+\beta_1, \ldots, u+\beta_{r-2}] \asymp cT/N^{r-1}$, we deduce

$$c \gg B_3 := \frac{d_1 N^{r-1}}{dTG^{(r-2)}(\beta_0, \beta_1, \ldots, \beta_{r-2})}.$$

Combining the cases that (i) holds or (ii) holds, we have shown that either $c \gg B_1$ or $B_2 \gg c \gg B_3$. Note that $B_2/B_1 \ll \delta A^{(r-1)(r-2)/2} \ll 1$ so that $B_2 < B_1$ if the constant c_2 in (3) is sufficiently small. Therefore, the elements of $T(a_1, \ldots, a_{r-1})$ are in $\ll (N/B_1)+1 \ll N/B_1$ intervals each having length $\ll B_2$. The minimal distance between two elements of $T(a_1, \ldots, a_{r-1})$ is $\gg B_3$. Hence,

$$t(a_1, \ldots, a_{r-1}) \ll \frac{N}{B_1}\left(\frac{B_2}{B_3} + 1\right).$$

The number of $u \in S'$ for which $u, u+a_1, \ldots, u+a_1+\cdots+a_{r-1}$ are consecutive elements of S' and $\max_{1\leq j\leq r-1}\{a_j\} > A$ is $\ll N/A$. If we consider the elements of S' other than these $O(N/A)$ elements, each belongs to some $T(a_1, \ldots, a_{r-1})$ with $\max_{1\leq j\leq r-1}\{a_j\} \leq A$. Therefore,

$$|S'| \ll \sum_{a_1 \leq A} \cdots \sum_{a_{r-1} \leq A} t(a_1, \ldots, a_{r-1}) + \frac{N}{A}.$$

Using the estimates

$$G^{(r-1)}(\beta_0,\ldots,\beta_{r-1}) \ll A^{r(r-1)/2}, \quad \frac{G^{(r-2)}(\beta_0,\ldots,\beta_{r-2})}{d_1} \ll A^{(r-2)(r-3)/2},$$

and

$$|M_j^{(r-1)}(\beta_0,\ldots,\beta_{r-1})| \ll A^{(r-1)(r-2)/2} \quad \text{for } 0 \le j \le r-1,$$

we obtain

$$\frac{N}{B_1}\left(\frac{B_2}{B_3}+1\right) \ll \left(\frac{\delta T}{N^{r-2}} A^{r^2-4r+4} + \frac{T}{N^{r-1}} A^{(r^2-r)/2}\right).$$

Therefore,

$$|S'| \ll \frac{\delta T}{N^{r-2}} A^{r^2-3r+3} + \frac{T}{N^{r-1}} A^{(r^2+r-2)/2} + \frac{N}{A}.$$

Finally, we take

$$A = \min\left\{N^{2/(r+1)} T^{-2/(r^2+r)}, c_2 \delta^{-2/((r-1)(r-2))}, \left(\delta T N^{1-r}\right)^{-1/(r^2-3r+4)}\right\}$$

and use (ii) of Lemma 5 to obtain the estimate in the statement of Theorem 6. ■

Theorem 6 is a variation of Theorem 1 of Huxley's paper [22] and of the main result in the paper of Huxley and Sargos [24]. Both of these results are derived from the use of divided differences. The main new ingredient in our approach is in our use of the greatest common divisor defined in Lemma 3. On the other hand, their approaches seem to allow one to use larger values of δ. If needed, one could try to incorporate their ideas to take advantage of the wider range of δ. We chose not to simply because any such improvement would not strengthen any of the applications given in the Introduction. For the purposes of the applications given in the Introduction, we should mention that the result of Huxley and Sargos in [24] can be used to improve on the first author's work in [7] but does not lead to a result as strong as Theorem 4. Also, to obtain Theorem 3, we make additional use of another result by Huxley in [22] which is of a similar nature to Theorem 6.

Our main aim in establishing Theorem 6 was to allow what we view as the main term in our estimate for $|S|$ to also be the dominant term for our applications. We view the main term as $T^{2/(r^2+r)} N^{(r-1)/(r+1)}$. There are other variations of Theorem 6 that can be obtained by using some parts of the proof and not other parts. For example, under the same conditions as Theorem 6 but with

$$\delta < kTN^{-(2r^2-3r-3)/(2r-3)},$$

one can obtain that

$$|S| \ll T^{2/(r^2+r)} N^{(r-1)/(r+1)} + N\left(\delta T N^{1-r}\right)^{1/(r^2-3r+4)} + N^{(2r-4)/(2r-3)}.$$

This estimate would suffice for establishing Theorem 3 but would lead to a weaker result than Theorem 4. Theorem 6 as it stands will enable us to derive both Theorem 3 and Theorem 4. Another estimate of this type which follows from our techniques is

$$|S| \ll T^{2/(r^2+r)} N^{(r-1)/(r+1)} + \delta^{2/(r^2-r)} N,$$

which holds for any δ. This last estimate is the main result of Huxley and Sargos in [24].

3. A Second Result on Fractional Parts

The main result in the previous section involved fairly general conditions on the function f under consideration. As we mentioned at the beginning of that section, the applications discussed in the introduction will only make use of the results of this section and the previous in the case that $f(u) = X/u^s$ where X is a new parameter and where s is a positive rational number. This section will involve restricting the function f to functions of the form $f(u) = X/u^s$ with s rational. We will make use of the following

Lemma 8. *Let ℓ be a positive integer, and let $s \in \mathbb{Q} - \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$. Then there exist homogeneous polynomials $P(u,a)$ and $Q(u,a)$ in $\mathbb{Z}[u,a]$ of degree ℓ with $P(0,a) \not\equiv 0$ and $Q(0,a) \not\equiv 0$ and real numbers $w_1(\ell,s), w_2(\ell,s), \dots$ with $w_1(\ell,s) \neq 0$ such that if $|a| < u$, then*

$$(u+a)^s P - u^s Q = \sum_{j=1}^{\infty} w_j(\ell,s) a^{2\ell+j} u^{s-\ell-j}. \tag{12}$$

Proof. The lemma is a consequence of known results about Padé approximations. We refer to [1] to justify the lemma. There is no harm in relaxing the condition that $P(u,a)$ and $Q(u,a)$ are in $\mathbb{Z}[u,a]$ to the condition that $P(u,a)$ and $Q(u,a)$ are in $\mathbb{Q}[u,a]$, so we only concern ourselves with constructing $P(u,a)$ and $Q(u,a)$ in $\mathbb{Q}[u,a]$. For real numbers α, β, and γ, with $\gamma \notin \{0,-1,-2,\dots\}$, we consider the hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j(\beta)_j}{(\gamma)_j(1)_j} z^j$$

where $(x)_0 = 1$ and $(x)_j = x(x+1)(x+2)\cdots(x+j-1)$ for j a positive integer. In Chapter 2 of [1], the authors describe how to find explicit rational functions which approximate the hypergeometric function

$${}_2F_1(\alpha,1;\gamma;z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j} z^j.$$

We take $\alpha = -s$ and $\gamma = 1$ and note that

$$(1-z)^s = {}_2F_1(-s,1;1;z).$$

In Theorem 1.1.1 of [1], we take $L = M = \ell$ and obtain that there exist polynomials $p(z)$ and $q(z)$ in $\mathbb{Q}[z]$ each of degree $\le \ell$ such that formally

$$p(z)(1-z)^s - q(z) = \sum_{j=1}^{\infty} d_{2\ell+j} z^{2\ell+j} \tag{13}$$

for some real numbers $d_{2\ell+j}$. Note that since $p(z)$ and $q(z)$ are polynomials, the series converges in $\{z : |z| < 1\}$. From (1.6) and (1.9) of Chapter 2 of [1], we deduce

$$p(z) = \left(\prod_{j=1}^{\ell} \frac{(-s)_j}{j!}\right) \frac{\prod_{j=1}^{\ell-1}\left((-s-j)^{\ell-j}(\ell-j)!\right)}{\prod_{k=1}^{\ell-1}\prod_{j=k}^{\ell-1}(\ell-j+2k-1)_2} \sum_{j=0}^{\ell} \frac{(-\ell)_j(s-\ell)_j}{(-2\ell)_j(1)_j} z^j. \tag{14}$$

We take $z = -a/u$. Then from (13) we deduce

$$p\left(\frac{-a}{u}\right)(u+a)^s - q\left(\frac{-a}{u}\right)u^s = \sum_{j=1}^{\infty} d_{2\ell+j}\left(\frac{-a}{u}\right)^{2\ell+j} u^s.$$

Multiplying through by u^ℓ and setting

$$P(u,a) = u^\ell p\left(\frac{-a}{u}\right) \quad \text{and} \quad Q(u,a) = u^\ell q\left(\frac{-a}{u}\right),$$

we can rewrite the above in the form given in (12). Clearly, $P(u,a)$ and $Q(u,a)$ are in $\mathbb{Q}[u,a]$. It follows easily from (14) and the conditions in the lemma on s that the coefficients of $p(z)$ and, hence, $P(u,a)$ are all non-zero so that $P(u,a)$ is a polynomial of degree ℓ and $P(0,a) \not\equiv 0$. The leading coefficient of $q(z)$ and $w_1(\ell,s)$ can also be given explicitly. From (1.9) of Chapter 1 of [1], the coefficient of z^ℓ in $q(z)$ is $(-1)^\ell$ times the determinant of the $(\ell+1)\times(\ell+1)$ matrix where the entry in the ith row and jth column is $(-s)_{i+j-2}/(1)_{i+j-2}$. From (1.3) and (1.6) of Chapter 2 of [1] (with $L = \ell$ and $M = \ell+1$), we see that this coefficient is

$$(-1)^\ell \left(\prod_{j=1}^{\ell+1} \frac{(-s)_{j-1}}{(j-1)!}\right) \frac{\prod_{j=1}^{\ell}\left((-s-j)^{\ell+1-j}(\ell+1-j)!\right)}{\prod_{k=1}^{\ell}\prod_{j=k}^{\ell}(\ell-j+2k-1)_2}.$$

Therefore, the polynomial $q(z)$ and, hence, $Q(u,a)$ is of degree ℓ and $Q(0,a) \not\equiv 0$. Similarly, from (1.11) of Chapter 1 and (1.3) and (1.6) of Chapter 2 of [1], we can deduce an explicit formula for $w_1(\ell,s)$, noting in particular that $w_1(\ell,s) \neq 0$. ■

Observe that the series on the right-hand side of (13) converges absolutely for z with $|z| < 1$. It easily follows that if $|a| < u/2$ and if r is any positive integer, then

$$(u+a)^s P - u^s Q = \sum_{j=1}^{r-1} w_j(\ell,s) a^{2\ell+j} u^{s-\ell-j} + O_{\ell,s}\left(a^{2\ell+r}u^{s-\ell-r}\right).$$

When referring to Lemma 8, we will make use of the above equation.

Lemma 8 is similar to work done by Huxley and Nair in [23]. It is perhaps interesting to note that a different approach to establishing the lemma can be given in the case that s is an integer (which it will be in our applications based on the results in this section); see the first author's [8]. We have chosen the present approach because it is reasonable to expect future applications when s is not an integer.

Theorem 7. *Let k be an integer ≥ 2, and let $s \in \mathbb{Q} - \{-(k-1), -(k-2), \ldots, k-2, k-1\}$. Let δ and N be positive real numbers. Let $f(u) = X/u^s$, where X is an arbitrary real number independent of u and δ but possibly depending on k, N, and s. Suppose that*

$$N^s \leq X \quad \textit{and} \quad \delta \leq c_3 N^{-(k-1)}$$

where $c_3 = c_3(k,s) > 0$ is sufficiently small. Set

$$S = \{u \in \mathbb{Z} \cap (N, 2N] : ||f(u)|| < \delta\}.$$

Then

$$|S| \ll_{k,s} X^{1/(2k+1)} N^{(k-s)/(2k+1)} + \delta X^{1/(6k+3)} N^{(6k^2+2k-s-1)/(6k+3)}.$$

Throughout the remainder of this section, constants may depend on k and s but are independent of X, N, and δ. To establish the result above, we make use of a lemma which is similar to Lemma 5.

Lemma 9. *For some positive constant c_4, independent of c_3, there exists a set S' satisfying*
(i) $S' \subseteq S$,
(ii) $|S| \leq 2k(|S'| + 1)$,
and
(iii) any two distinct elements of S' differ by more than $c_4 X^{-1/(2k-1)} N^{(k+s)/(2k-1)}$.

Proof. Suppose that u, $u+a$, and $u+a+b$ are all in S with a and b positive integers $\leq c_4 X^{-1/(2k-1)} N^{(k+s)/(2k-1)}$. We will choose c_4 to be sufficiently small. We view u and a as fixed and b as a variable. We will show that there are $\leq 2k-2$ choices for b. The lemma will follow by choosing the elements of S' to be every $(2k)$th element of S.

Observe that $k \geq 2$ and $N^s \leq X$ imply that a and b are small compared to u. There are integers m_1, m_2, and m_3 such that $f(u) = m_1 + O(\delta)$, $f(u+a) = m_2 + O(\delta)$, and $f(u+a+b) = m_3 + O(\delta)$. In fact, each of the m_j are positive since $f(t) \geq 2^{-|s|} X N^{-s} \geq 2^{-|s|}$ for $t \in (N, 2N]$ and $\delta \leq c_3 N^{-(k-1)} \leq c_3$. For ℓ a positive integer $\leq k-1$, we get by Lemma 8 that there are homogeneous polynomials $P_\ell(u,a)$ and $Q_\ell(u,a)$ in $\mathbb{Z}[u,a]$ of degree ℓ with $P_\ell(0,a) \not\equiv 0$ and $Q_\ell(0,a) \not\equiv 0$ such that

$$P_\ell(u,a)f(u) - Q_\ell(u,a)f(u+a) = X \frac{P_\ell(u,a)(u+a)^s - Q_\ell(u,a)u^s}{u^s(u+a)^s} \tag{15}$$

$$= \frac{X}{u^s(u+a)^s} \sum_{j=1}^{\infty} w_j(\ell,s) \frac{a^{2\ell+j}}{u^{-s+\ell+j}},$$

where $w_1(\ell,s) \neq 0$. We consider the case that $\ell = k-1$. Since c_4 is sufficiently small, the bound on a above implies that the right-hand side of (15) is $< 1/2$. On the other hand, since P_{k-1} and Q_{k-1} are of degree $k-1$, we deduce that

$$P_{k-1}(u,a)f(u) - Q_{k-1}(u,a)f(u+a) = P_{k-1}(u,a)m_1 - Q_{k-1}(u,a)m_2 + O(\delta N^{k-1}).$$

Observe that the implied constant here depends only on P_{k-1} and Q_{k-1}; in particular, the implied constant does not depend on c_3. Now $\delta \leq c_3 N^{-(k-1)}$ allows us to consider the error term above to have absolute value $< 1/2$. Since $P_{k-1}(u,a)m_1 - Q_{k-1}(u,a)m_2$ is an integer, we now deduce that

$$P_{k-1}(u,a)m_1 - Q_{k-1}(u,a)m_2 = 0. \tag{16}$$

An analogous argument gives

$$P_{k-1}(u+a,b)m_2 - Q_{k-1}(u+a,b)m_3 = 0$$

and

$$P_{k-1}(u,a+b)m_1 - Q_{k-1}(u,a+b)m_3 = 0.$$

We multiply the first of these last two equations by $Q_{k-1}(u,a+b)$ and the second by $-Q_{k-1}(u+a,b)$ and add to obtain

$$(17) \qquad P_{k-1}(u+a,b)Q_{k-1}(u,a+b)m_2 - P_{k-1}(u,a+b)Q_{k-1}(u+a,b)m_1 = 0.$$

Recall that we are viewing u and a as being fixed so that m_1 and m_2 are fixed. Also, $P_{k-1}(0,b) \not\equiv 0$ and $Q_{k-1}(0,b) \not\equiv 0$. On the other hand, the left-hand side of (17) is a polynomial of degree $\le 2k-2$ in b and the term involving b^{2k-2} is $P_{k-1}(0,b)Q_{k-1}(0,b)(m_2 - m_1)$. Now, we claim that $m_2 - m_1 \ne 0$. Assume $m_2 - m_1 = 0$. Then (16) implies that $P_{k-1}(u,a) = Q_{k-1}(u,a)$ so that

$$\begin{aligned} P_{k-1}(u,a)f(u) - &Q_{k-1}(u,a)f(u+a) \\ &= Q_{k-1}(u,a)f(u) - Q_{k-1}(u,a)\left(f(u) + af'(u)(1+o(1))\right) \\ &\gg Q_{k-1}(u,a)aXN^{-(s+1)} \gg a^k XN^{-(s+1)}, \end{aligned}$$

where the implied constants are independent of c_4. The bound on a now implies that (15) with $\ell = k-1$ cannot hold, giving a contradiction. Thus, we get that $m_2 - m_1 \ne 0$ and (17) represents a polynomial in b of degree $2k-2$. This gives that there are $\le 2k-2$ positive integers b as above, completing the proof of the lemma. ∎

We set

$$R = c_4 X^{-1/(2k-1)} N^{(k+s)/(2k-1)}.$$

It follows from $X \ge N^s$ and Lemma 9 that

$$|S| \ll \frac{N}{R} + 1 \ll X^{1/(2k-1)} N^{(k-s-1)/(2k-1)} \ll X^{1/(2k+1)} N^{(k-s)/(2k+1)} \quad \text{if } X \le N^{(2s+1)/2}.$$

Therefore, to establish Theorem 7, we need only consider $X > N^{(2s+1)/2}$, and we do so.

For each positive integer a, we define

$$T(a) = \{u : u \text{ and } u+a \text{ are consecutive elements in } S'\}$$

and

$$t(a) = |T(a)|.$$

Then

$$|S'| \le 1 + \sum_{a=1}^{\infty} t(a).$$

Condition (iii) of Lemma 9 implies

$$\sum_{a=1}^{\infty} t(a) = \sum_{a>R} t(a).$$

Observe that for any A, we also have that

$$N \geq \sum_{a=1}^{\infty} at(a) \geq A \sum_{a>A} t(a)$$

so that

$$\sum_{a>A} t(a) \leq N/A.$$

We will obtain an estimate for $\sum_{R<a\leq A} t(a)$, and then choose A appropriately. To help achieve such an estimate we establish the following result.

Lemma 10. *Let $a \in (R, A]$ with*

$$A \leq N^{2/3}. \tag{18}$$

Let $I \subseteq (N, 2N]$ with

$$|I| \leq c_5 X^{-1} N^{k+s+1} a^{-(2k-1)},$$

where c_5 is a sufficiently small positive constant. Of every 3 consecutive elements in $T(a) \cap I$, there are 2 consecutive elements that differ by

$$\gg X^{-1/3} N^{(k+s+1)/3} a^{-(2k-3)/3}.$$

Proof. Suppose that u and two additional elements $u+b$ and $u+b+d$ are in $T(a) \cap I$ with b and d positive. Since $a \in (R, A]$, we obtain from (18) that $a \leq N^{2/3}$. Also, $a > R$ so that the bound on $|I|$ in the lemma is smaller than any given constant multiple of N. We deduce that

$$\max\{b, d\} \leq c_6 N,$$

where $c_6 > 0$ is sufficiently small. Let m_1 and m_2 be as defined in the proof of Lemma 9, and let m_4, m_5, m_6, and m_7 be integers with $f(u+b) = m_4 + O(\delta)$, $f(u+a+b) = m_5 + O(\delta)$, $f(u+b+d) = m_6 + O(\delta)$, and $f(u+a+b+d) = m_7 + O(\delta)$. We define

$$F(u,a) = P_{k-2}(u,a) f(u) - Q_{k-2}(u,a) f(u+a),$$

$$W = W(u,a,b,d) = dF(u,a) - (b+d)F(u+b,a) + bF(u+b+d,a),$$

and

$$\begin{aligned} V = d\,&(P_{k-2}(u,a)m_1 - Q_{k-2}(u,a)m_2) \\ &- (b+d)\,(P_{k-2}(u+b,a)m_4 - Q_{k-2}(u+b,a)m_5) \\ &+ b\,(P_{k-2}(u+b+d,a)m_6 - Q_{k-2}(u+b+d,a)m_7)\,. \end{aligned}$$

From (15) with $\ell = k-2$, we can deduce that

$$\frac{\partial^j F}{\partial u^j}(u,a) \asymp a^{2k-3} X N^{-(s+k-1+j)} \quad \text{for } j \in \{0,1,2,3\}.$$

We may view $W/(bd(b+d))$ as a second order divided difference of the function $F(u,a)$. The Mean Value Theorem and Lemma 2 now imply

$$W \asymp bd(b+d)\frac{\partial^2 F}{\partial u^2}(u,a).$$

Note that since u and $u+a$ are consecutive elements of S' and $u+b$ is in S', then $b \geq a > R$. Similarly, $d \geq a > R$. We deduce that

$$|W| \asymp a^{2k-3} bd(b+d) X N^{-(k+s+1)} \gg R^{2k-1}(b+d) X N^{-(k+s+1)} \gg (b+d)N^{-1},$$

where we note that this last implied constant depends on c_4. Since $P_{k-2}(u,a)$ and $Q_{k-2}(u,a)$ are homogeneous polynomials of degree $k-2$, we get that

$$P_{k-2}(u,a)m_1 - Q_{k-2}(u,a)m_2 = F(u,a) + O(\delta N^{k-2}),$$

$$P_{k-2}(u+b,a)m_4 - Q_{k-2}(u+b,a)m_5 = F(u+b,a) + O(\delta N^{k-2}),$$

and

$$P_{k-2}(u+b+d,a)m_6 - Q_{k-2}(u+b+d,a)m_7 = F(u+b+d,a) + O(\delta N^{k-2}).$$

It follows that

$$V = W + O\left((b+d)\delta N^{k-2}\right).$$

Note that this last implied constant is independent of δ and, hence, c_3. By the lower bound for $|W|$ above and the fact that $\delta \leq c_3 N^{-(k-1)}$, we get that (with c_3 sufficiently small compared to c_4) the error term in this last expression is smaller in absolute value than $|W|/2$. In particular, this means that $V \neq 0$. On the other hand, by definition, V is an integer. Thus, $(3/2)|W| \geq 1$ so that

$$a^{2k-3} bd(b+d) \gg X^{-1} N^{k+s+1}.$$

It follows that one of b or d is $\gg X^{-1/3} N^{(k+s+1)/3} a^{-(2k-3)/3}$, completing the proof. ■

Proof of Theorem 7. We choose

$$A = X^{-1/(2k+1)} N^{(k+s+1)/(2k+1)}.$$

Observe that (18) holds since $X \geq N^s$. Let $I \subseteq (N, 2N]$ as in Lemma 10. We will obtain an upper bound for $|T(a) \cap I|$. We denote 2 elements of $T(a) \cap I$ by u and $u+b$. In particular, the length of I is an upper bound for b. With m_1, m_2, m_4, and m_5 as before, we define

$$G(u,a) = P_{k-1}(u,a)f(u) - Q_{k-1}(u,a)f(u+a),$$

$$W' = G(u,a) - G(u+b,a),$$

and

$$V' = (P_{k-1}(u,a)m_1 - Q_{k-1}(u,a)m_2) - (P_{k-1}(u+b,a)m_4 - Q_{k-1}(u+b,a)m_5).$$

One gets from (15) that

$$W' \asymp \frac{a^{2k-1}bX}{N^{k+s+1}},$$

where the implied constant is independent of c_5. Also,

$$V' = W' + O\left(\delta N^{k-1}\right).$$

Since c_5 is sufficiently small and

$$b \le |I| \le c_5 X^{-1} N^{k+s+1} a^{-(2k-1)},$$

we get that both of the terms in the expression for V' above have absolute value $< 1/2$ so that $V' = 0$. Thus, $W' \ll \delta N^{k-1}$ so that $b \ll \delta X^{-1} N^{2k+s} a^{-(2k-1)}$. In other words, there is a sub-interval J of I with

$$|J| \ll \delta X^{-1} N^{2k+s} a^{-(2k-1)}$$

such that u and $u+b$ are in J. Recall that u and $u+b$ are any 2 elements of $T(a) \cap I$. From Lemma 10, of every 3 consecutive elements in $T(a) \cap I$, there are 2 consecutive elements that differ by

$$\gg X^{-1/3} N^{(k+s+1)/3} a^{-(2k-3)/3}.$$

We deduce

$$|T(a) \cap I| \ll \frac{|J|}{X^{-1/3} N^{(k+s+1)/3} a^{-(2k-3)/3}} + 1 \ll \delta X^{-2/3} N^{(5k+2s-1)/3} a^{-4k/3} + 1.$$

From the beginning of the proof of Lemma 10, the upper bound on $|I|$ given in Lemma 10 is $< N$. Accounting for the number of such intervals I needed to obtain all of $(N, 2N]$, we get that

$$|T(a)| \ll \frac{N}{X^{-1} N^{k+s+1} a^{-(2k-1)}} \left(\delta X^{-2/3} N^{(5k+2s-1)/3} a^{-4k/3} + 1\right)$$

$$\ll \delta X^{1/3} N^{(2k-s-1)/3} a^{(2k-3)/3} + X N^{-(k+s)} a^{2k-1}.$$

Thus,

$$\sum_{R<a\le A} t(a) \ll \delta X^{1/3} N^{(2k-s-1)/3} \sum_{1\le a\le A} a^{(2k-3)/3} + XN^{-(k+s)} \sum_{1\le a\le A} a^{2k-1}$$

$$\ll \delta X^{1/3} N^{(2k-s-1)/3} A^{2k/3} + XN^{-(k+s)} A^{2k}.$$

Our choice of A now gives

$$|S| \le 1 + \sum_{a=1}^{\infty} t(a) \ll \delta X^{1/(6k+3)} N^{(6k^2+2k-s-1)/(6k+3)} + X^{1/(2k+1)} N^{(k-s)/(2k+1)}.$$

This is the desired result. ∎

4. Gaps Between k−free Numbers

In this section, we show how to establish Theorem 1 from Theorem 7. Fix an integer $k \ge 2$. To establish both parts of Theorem 1, it suffices to consider $h = x^{1/(2k+1)}(\log x)g(x)$ where either $g(x) = c$ with c a sufficiently large constant or $g(x)$ is an increasing function which tends to infinity with x and $g(x) \le \log x$ for all x. All constants (implied or otherwise) other than c are independent of c. We also consider x to be a sufficiently large real number. The letter p shall be used for primes. We set $z = \log\log x$. Then a simple sieve of Eratosthenes argument gives that the number of integers in $(x, x+h]$ not divisible by p^k for some $p \le z$ is

$$h\prod_{p\le z}\left(1-\frac{1}{p^k}\right) + O(\log x) = h\prod_{p}\left(1-\frac{1}{p^k}\right) + O(h/z) + O(\log x) = \frac{h}{\zeta(k)} + o(h).$$

It therefore suffices to show that the number of integers in $(x, x+h]$ divisible by p^k for some $p > z$ is less than εh for any prescribed $\varepsilon = \varepsilon(k)$ in the case $g(x) = c$ and is $o(h)$ in the case $g(x)$ increases to infinity.

The number of integers in $(x, x+h]$ divisible by p^k for some $p > z$ is bounded above by

$$\sum_{z<p\le 2x^{1/k}} M_p,$$

where for any integer u, M_u denotes the number of integers in $(x, x+h]$ divisible by u^k. We use that $\lim_{x\to\infty} \pi(x)/(x/\sqrt{\log x}) = 0$ where $\pi(x)$ is the number of primes $\le x$ (an easy consequence of a Chebyshev estimate for $\pi(x)$). We break up the sum above into 2 sums $\sum_1$ and $\sum_2$, where

$$\sum\nolimits_1 = \sum_{z<p\le h\sqrt{\log x}} M_p$$

$$\le \sum_{z<p\le h\sqrt{\log x}} \left(\left[\frac{h}{p^k}\right]+1\right) \le \sum_{p>z}\frac{h}{p^2} + O\left(\pi\left(h\sqrt{\log x}\right)\right) = o(h)$$

and

$$\sum\nolimits_2 = \sum_{h\sqrt{\log x}<p\le 2x^{1/k}} M_p \le \sum_{h\sqrt{\log x}<u\le 2x^{1/k}} M_u,$$

with u running over the integers rather than the primes in the last sum. We now need only show that this last sum is less than $\varepsilon h/2$ in the case $g(x)=c$ and is $o(h)$ in the case $g(x)$ increases to infinity.

Note that with $u>(h\sqrt{\log x})/2$, we get that $M_u=0$ or 1 and that $M_u=1$ precisely when there is an integer m such that $mu^k\in(x,x+h]$. Let

$$S(t_1,t_2)=\{u\in(t_1,t_2]: M_u=1\}.$$

Observe that

$$\sum_{h\sqrt{\log x}<u\le 2x^{1/k}} M_u \le \sum_{j=0}^{r} |S(2^{-j}x^{1/k}, 2^{-j}2x^{1/k})| \tag{19}$$

where r is chosen so that

$$2^r \ge \frac{x^{1/k}}{h\sqrt{\log x}} > 2^{r-1}.$$

We consider u in $S(N,2N)$ where $h\sqrt{\log x}\le 2N\le 2x^{1/k}$. Let m denote the integer for which $mu^k\in(x,x+h]$. Then

$$\frac{x}{u^k}<m\le\frac{x}{u^k}+\frac{h}{u^k}.$$

We set $f(u)=x/u^k$ and $\delta=2hN^{-k}$. Then

$$||f(u)||<\delta.$$

We take $X=x$ and $s=k$ and apply Theorem 7 to deduce that

$$|S(N,2N)| \ll x^{1/(2k+1)} + hx^{1/(6k+3)}N^{-1/3}.$$

By considering $N=2^{-j}x^{1/k}$ and summing in (19), we deduce that

$$\sum_{h\sqrt{\log x}<u\le 2x^{1/k}} M_u \ll x^{1/(2k+1)}\log x + \frac{h}{\sqrt{\log x}}.$$

For c sufficiently large and $g(x)=c$, we obtain $\sum_2 \le \varepsilon h/2$, and for $g(x)$ increasing to infinity, we obtain $\sum_2 = o(h)$. Thus, Theorem 1 follows.

5. The Gap Problem for Finite Abelian Groups

In this section, we establish Theorem 2 using Theorem 1 (with $k = 2$) and, hence, using indirectly Theorem 7. Thus, we are interested in the problem of determining an $h = h(x)$ as small as possible such that for every sufficiently large x, the interval $(x, x + h]$ contains $P_k h + o(h)$ elements from A_k for some constant P_k. In the case that $k = 1$, the set A_k is the set of squarefree numbers so one cannot in general expect to obtain a result better than $h = x^{1/5}(\log x)g(x)$, for every function $g(x)$ increasing to infinity, without improving on Theorem 1. The main purpose of this section is to show that, in fact, the general problem here is equivalent to the gap problem for squarefree numbers. Observe, however, that Theorem 8 below is sufficient for establishing Theorem 2.

In Theorems 8 and 9 below, we make use of intervals of the form $(x, x + h]$ with a condition that $h \geq cx^\theta$ for some constant c; we note here that this is done for convenience and the role of cx^θ may be replaced by $x^\theta \log x$ or $x^\theta g(x)$ for any function $g(x)$ increasing to infinity.

Theorem 8. *Let $\theta \in (0, 1)$. Suppose that there is a constant $c_7 > 0$ such that if h is a function of x with $h \geq c_7 x^\theta$, then the number of squarefree numbers in the interval $(x, x + h]$ is*

$$\frac{6}{\pi^2}h + o(h)$$

as x approaches infinity. Let $w(n)$ be a multiplicative function for which $w(p) = 1$ for every prime p, and let $g(x)$ be an arbitrary function which tends to infinity with x. Let k be a positive integer. If $h = x^\theta g(x)$, then the number of integers n in the interval $(x, x+h]$ for which $w(n) = k$ is

$$P_k h + o(h)$$

where P_k depends only on k and w.

To prove the theorem, we will consider a number $\epsilon > 0$. Unless stated otherwise, all implied constants below may depend on k and w but will be independent of ϵ. When referring to h, we shall suppose $h = x^\theta g(x)$ with $g(x)$ as in the theorem.

Lemma 11. *Let $\epsilon > 0$, and let $N = N(\epsilon)$ be a sufficiently large positive integer. Let*

$$S_2 = \{n \in (x, x + h] : \text{ there is a } p \leq N \text{ such that } p^N | n\}.$$

Then $|S_2| \ll \epsilon h$.

Proof. Since

$$|S_2| \leq \sum_{p \leq N} \left(\left[\frac{x+h}{p^N} \right] - \left[\frac{x}{p^N} \right] \right) \leq \sum_{p \leq N} \left(\frac{h}{p^N} + 1 \right) = \left(\sum_{p \leq N} \frac{1}{p^N} \right) h + O(N) \leq \frac{Nh}{2^N} + O(N),$$

the lemma easily follows. ■

Lemma 12. *Suppose the conditions in Theorem 8 hold. Let $\epsilon > 0$. Let $N = N(\epsilon)$ be a sufficiently large positive integer. Set*

$$P = P(N) = \prod_{p \le N} p^N \quad \text{and} \quad Q = Q(N) = \prod_{p \le N} p^{N-1}.$$

Let

$$S_1 = \{n \in (x, x+h] : p^2 | n \text{ for some } p > N, \text{ and } \gcd(n, P) | Q\}.$$

Then $|S_1| \ll \epsilon h$.

Proof. Observe that every integer $n \in S_1$ can be written in the form ab where a and b are relatively prime integers such that (i) if $p \le N$ and $p|a$, then $p^2 \nmid a$, and (ii) $b|Q$ and b is squarefull. It follows that if $n \in S_1$, then $n = ab \in (x, x+h]$ for some relatively prime integers a and b as in (i) and (ii), with some prime $p > N$ satisfying $p^2|a$. For such a and b, we have that a is in the interval $I = ((x/b), (x/b) + (h/b)]$. From the conditions in Theorem 8, we deduce that the number of squarefree numbers in I is $6h/(b\pi^2) + o(h/b)$. On the other hand, a simple sieve argument gives that the number of integers in I which are not divisible by p^2 for every prime $p \le N$ is

$$\prod_{p \le N} \left(1 - \frac{1}{p^2}\right) \frac{h}{b} + O\left(2^N\right) = \frac{6h}{\pi^2 b} + O\left(\frac{h}{bN}\right) + O\left(2^N\right).$$

Therefore, the number of integers a in I which satisfy (i) and which are divisible by the square of a prime $> N$ is

$$O\left(\frac{h}{bN}\right) + O\left(2^N\right) + o(h/b).$$

Since

$$\sum_{b \text{ squarefull}, \, b|Q} \frac{1}{b} = \prod_{p \le N} \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^{N-1}}\right) \le \prod_{p \le N} \left(1 + \frac{1}{p(p-1)}\right) \ll 1$$

where the implied constant is absolute, we deduce

$$|S_1| \ll O\left(\frac{h}{N}\right) + O\left(2^N Q\right) + o(h) \ll \epsilon h.$$

The lemma now follows. ∎

Proof of Theorem 8. Let B be the set of integers n for which $w(n) = k$. Let $\epsilon > 0$, and define N as in Lemma 11 and Lemma 12. Consider the set T of integers n in the interval $(x, x+h]$ that satisfy

(20) $n \equiv md \pmod{P}$ for some $d \in B$ and $m \in \mathbb{Z}$ such that $d|Q$ and $\gcd(m, Q) = 1$.

Thus, $n \in (x, x+h]$ is in T if and only if for some $d \in B$ dividing Q we have $d|n$ and the prime divisors of n/d are $> N$. Let $\alpha = \alpha(N)$ denote the number of $md \in [1, P]$ as in (20).

Then the number of such $n \in T$ is $\alpha h/P + O(P)$. Since $w(p) = 1$ for all primes p and w is multiplicative, we have that if $n \in T$ and $n \notin B$, then $n \in S_1$. On the other hand, if $n \notin T$ and $n \in B$, then $n \in S_1 \cup S_2$. In other words,

$$|\{n \in (x, x+h] : n \in B\}| = \frac{\alpha(N)}{P(N)} h + O(P(N)) + O(|S_1|) + O(|S_2|).$$

Since N is fixed, it is clear that $P(N) \ll \epsilon h$. Thus, Lemma 11 and Lemma 12 imply

$$|\{n \in (x, x+h] : n \in B\}| = \frac{\alpha(N)}{P(N)} h + O(\epsilon h).$$

It follows that

$$\limsup_{x\to\infty} \frac{|\{n \in (x, x+h] : n \in B\}|}{h} = \frac{\alpha(N)}{P(N)} + O(\epsilon)$$

and

$$\liminf_{x\to\infty} \frac{|\{n \in (x, x+h] : n \in B\}|}{h} = \frac{\alpha(N)}{P(N)} + O(\epsilon).$$

The above holds for any $\epsilon > 0$ and any sufficiently large fixed integer N. Therefore,

$$\limsup_{x\to\infty} \frac{|\{n \in (x, x+h] : n \in B\}|}{h} - \liminf_{x\to\infty} \frac{|\{n \in (x, x+h] : n \in B\}|}{h} = 0$$

so that $\lim_{x\to\infty}(|\{n \in (x, x+h] : n \in B\}|/h)$ exists, and the theorem follows. ■

Theorem 9. *Let w be a multiplicative function satisfying*

(i) the range of w is contained in the positive integers,

(ii) $w(p) = 1$ if p is a prime,

(iii) $w(p^{e+1}) > w(p^e)$ for every prime p and every positive integer e,

(iv) $w(p^e)$ where p is a prime and e is a positive integer depends only on e and not on p, and

(v) the set of primes dividing some value of $w(n)$ as n varies over the positive integers is infinite.

Let $\theta \in (0,1)$. Suppose there exists a k_0 and constants $c_8 = c_8(k) > 0$ and P_k such that if k is a positive integer $\geq k_0$ and h is a function of x for which $h \geq c_8 x^\theta$ for all x sufficiently large, then the number of integers $n \in (x, x+h]$ for which $w(n) = k$ is $P_k h + o(h)$. Then for every $h \geq 2c_8 x^\theta$, the number of squarefree numbers in $(x, x+h]$ is $(6/\pi^2)h + o(h)$.

Lemma 13. *Suppose w is a multiplicative function satisfying (i) through (v). Then there exist integers f and k with $\min\{f, k\}$ arbitrarily large such that if n is a positive integer for which $w(n) = k$, then n has a unique representation in the form ab where $a = p^f$ with p prime and where b is a squarefree number relatively prime to p.*

Proof. Conditions (iv) and (v) imply that for any fixed prime p, the set of primes dividing $w(p^e)$ as e varies over the positive integers is infinite. In particular, by considering a sufficiently large prime dividing some $w(p^e)$ and fixing f to be the minimal positive integer for which the prime divides $w(p^f)$, we get that there is an arbitrarily large f such that $w(p^f)$ is divisible by some prime which does not divide $\prod_{j<f} w(p^j)$. Let $k = w(p^f)$. Observe that if $p_1, \ldots, p_r$ are distinct primes and $e_1, \ldots, e_r$ are integers ≥ 2 for which $w(p_1^{e_1} \cdots p_r^{e_r}) = k$, then $w(p_1^{e_1}) \cdots w(p_r^{e_r}) = k$ so that the minimality condition on f and (i) through (iii) imply $r = 1$ and $e_1 = f$. In other words, if a is squarefull and $w(a) = k$, then $a = p^f$ for some prime p. The result of the lemma easily follows. ■

Lemma 14. *Let w be as in Theorem 9. Let $\epsilon > 0$. Let f and k be as in Lemma 13 with $(3/2)^f > 1/\epsilon$ and $k \geq k_0$. Then*

$$2^f P_k = \frac{4}{\pi^2} + O(\epsilon).$$

Proof. For $h \geq c_8 x^\theta$, the number of integers $n \in (x, x+h]$ for which $w(n) = k$ is $P_k h + o(h)$. In particular, this is true for $h = x$. A simple sieve argument gives that the number of integers in $(x, 2x]$ of the form $2^f m$ where m is an odd squarefree number is $(4/\pi^2)(x/2^f) + o(x)$. For each prime p, the number of integers in $(x, 2x]$ divisible by p^f is $\ll x/p^f$. Since f and k are as in Lemma 13, we deduce that the number of integers $n \in (x, 2x]$ for which $w(n) = k$ is

$$P_k x + o(x) = \frac{4}{2^f \pi^2} x + o(x) + O\left(\sum_{p \geq 3} \frac{x}{p^f}\right) = \frac{4}{2^f \pi^2} x + o(x) + O\left(\frac{x}{3^f}\right).$$

The result now follows from the inequality $(3/2)^f > 1/\epsilon$. ■

Lemma 15. *Under the conditions in Lemma 14 and with $f \geq 1/\theta$, if $h \geq c_8 x^\theta$, then the number of integers $n \in (2^f x, 2^f(x+h)]$ for which $w(n) = k$ and $2^f \nmid n$ is $\ll \epsilon h$.*

Proof. The condition $f \geq 1/\theta$ implies that if a multiple of p^f occurs in $(2^f x, 2^f(x+h)]$, then $p \leq 3x^{1/f} \leq 3x^\theta$. The number of integers in $(2^f x, 2^f(x+h)]$ divisible by p^f is $\ll 2^f h/p^f + 1$. Hence, the number of integers $n \in (2^f x, 2^f(x+h)]$ for which $w(n) = k$ and $2^f \nmid n$ is

$$\ll \sum_{3 \leq p \leq 3x^\theta} \left(\frac{2^f h}{p^f} + 1\right) \ll \frac{2^f h}{3^f} + \pi(3x^\theta).$$

Since $\pi(3x^\theta) = o(h)$, the result follows from the inequality $(3/2)^f > 1/\epsilon$. ■

Lemma 16. *Under the conditions in Lemma 15, if $h \geq c_8 x^\theta$, then the number of integers $n \in (2^f x, 2^f(x+h)]$ for which $w(n) = k$ and $2^f | n$ is $4h/\pi^2 + O(\epsilon h)$.*

Proof. By the conditions in Theorem 9 and by Lemma 14, the number of integers $n \in (2^f x, 2^f(x+h)]$ for which $w(n) = k$ is $2^f P_k h + o(h) = 4h/\pi^2 + O(\epsilon h)$. The result now follows from Lemma 15. ■

Lemma 17. *Let $h > 0$. The number of squarefree integers in $(x, x+h]$ is equal to the number of odd squarefree integers in $(x/2, (x+h)/2]$ plus the number of odd squarefree integers in $(x, x+h]$.*

The proof of Lemma 17 is immediate since each even squarefree number in $(x, x+h]$ is of the form $2n$ where $n \in (x/2, (x+h)/2]$, n is odd, and n is squarefree. We are now ready to establish Theorem 9.

Proof of Theorem 9. Let ϵ be an arbitrary number > 0, and choose f and k as above. The number of odd squarefree integers in $(x, x+h]$ is equal to the number of integers $n \in (2^f x, 2^f(x+h)]$ for which $w(n) = k$ and $2^f | n$. By Lemma 16, this is $4h/\pi^2 + O(\epsilon h)$. Observe that $h \geq 2c_8 x^\theta$ implies $h/2 \geq c_8 (x/2)^\theta$. We deduce from a second application of Lemma 16 that the number of odd squarefree integers in $(x/2, (x+h)/2]$ is $2h/\pi^2 + O(\epsilon h)$. The theorem follows from Lemma 17. ■

Corollary. *Let A_k be as defined in Theorem 2. Let $\theta \in (0,1)$, and let $k_0 \geq 1$. Then the following are equivalent:*

(i) For every function $u(x)$ which increases to infinity with x, the interval $(x, x+h]$ contains $(6/\pi^2)h + o(h)$ squarefree numbers, where $h = h(x) = x^\theta u(x)$.

(ii) For every integer $k \geq k_0$ and every function $u(x)$ which increases to infinity with x, the interval $(x, x+h]$ contains $P_k h + o(h)$ elements of A_k, where $h = h(x) = x^\theta u(x)$ and P_k is some constant depending only on k.

From a theorem of Schinzel (cf. [4], [43]), the unrestricted partition function $p(m)$ is such that the set

$$\{p : p \text{ is a prime dividing } p(m) \text{ for some positive integer } m\}$$

is infinite. The corollary follows from Theorem 8 and Theorem 9 upon noting that if $n = p_1^{e_1} \cdots p_r^{e_r}$ where $p_1, \ldots, p_r$ are distinct primes and where $e_1, \ldots, e_r$ are positive integers, then $a(n) = p(e_1)p(e_2)\cdots p(e_r)$ (cf. [20, p. 115]). As we noted at the beginning of this section, the role of x^θ may be replaced by $x^\theta \log x$ (as is the case when deriving Theorem 2 from Theorem 1) or by $x^\theta g(x)$ for any function $g(x)$ increasing to infinity with x.

6. Gaps Between Squarefull Numbers

In this section, we show how to obtain Theorem 3 by combining Theorem 6 with a result of Huxley [22]. An important contribution to this problem was made by Heath-Brown [19] and later observed independently by Liu [33]. We summarize this basic contribution with the following lemma.

Lemma 18. *Let x be sufficiently large, and let $\theta \in (0, 1/2)$. Let w be an arbitrary positive real number. Let $h = x^{(1/2)+\theta}$. Then for some sufficiently small $\delta = \delta(\theta) > 0$,*

$$Q(x+h) - Q(x) = \frac{\zeta(3/2)}{2\zeta(3)} x^\theta \left(1 + O\left(x^{-\delta}\right)\right) + O(S_1) + O(S_2)$$

where

$$S_1 = S_1(w) = \sum_{x^{\theta-\delta} < n \le w} \left(\left[\sqrt[3]{\frac{x+h}{n^2}} \right] - \left[\sqrt[3]{\frac{x}{n^2}} \right] \right)$$

and

$$S_2 = S_2(w) = \sum_{x^{\theta-\delta} < n \le 2x^{1/3}w^{-2/3}} \left(\left[\sqrt{\frac{x+h}{n^3}} \right] - \left[\sqrt{\frac{x}{n^3}} \right] \right),$$

where in the definition of S_1 and S_2, $[*]$ represents the greatest integer function.

Proof. Observe that every squarefull number can be represented uniquely in the form a^2b^3 with b squarefree. Furthermore, if $a^2b^3 \le x+h \le 2x$, then either $a \le w$ or $b \le 2x^{1/3}w^{-2/3}$. The number of squarefull numbers of the form $a^2b^3 \in (x, x+h]$ with $a \le x^{\theta-\delta}$ is

$$\begin{aligned}
\sum_{\substack{x < a^2b^3 \le x+h \\ a \le x^{\theta-\delta},\, b \text{ squarefree}}} 1 &\le \sum_{a \le x^{\theta-\delta}} \sum_{x/a^2 < b^3 \le (x+h)/a^2} 1 \\
&\le \sum_{a \le x^{\theta-\delta}} \left(\left[\sqrt[3]{\frac{x+h}{a^2}} \right] - \left[\sqrt[3]{\frac{x}{a^2}} \right] \right) \\
&\le \sum_{a \le x^{\theta-\delta}} \left(\frac{(x+h)^{1/3} - x^{1/3}}{a^{2/3}} \right) + O\left(x^{\theta-\delta}\right) \\
&\ll \sum_{a \le x^{\theta-\delta}} \left(\frac{hx^{-2/3}}{a^{2/3}} \right) + x^{\theta-\delta} \\
&\ll hx^{-2/3}\left(x^{\theta-\delta}\right)^{1/3} + x^{\theta-\delta} \\
&\ll x^{\theta-\delta}.
\end{aligned}$$

It follows that the number of squarefull numbers of the form $a^2b^3 \in (x, x+h]$ with $a \le w$ is $O(S_1) + O\left(x^{\theta-\delta}\right)$. Note that this also serves as an upper bound on the number of squarefull numbers of the form $a^2b^3 \in (x, x+h]$ for which both $a \le w$ and $b \le 2x^{1/3}w^{-2/3}$ hold.

Now, we combine S_2 with an estimate for the number of squarefull numbers of the form $a^2b^3 \in (x, x+h]$ with $b \le x^{\theta-\delta}$. Observe

$$\begin{aligned}
\sum_{\substack{x < a^2b^3 \le x+h \\ b \le x^{\theta-\delta},\, b \text{ squarefree}}} 1 &= \sum_{\substack{b \le x^{\theta-\delta} \\ b \text{ squarefree}}} \sum_{x/b^3 < a^2 \le (x+h)/b^3} 1 \\
&= \sum_{\substack{b \le x^{\theta-\delta} \\ b \text{ squarefree}}} \left(\frac{(x+h)^{1/2} - x^{1/2}}{b^{3/2}} \right) + O\left(x^{\theta-\delta}\right) \\
&= \sum_{\substack{b \le x^{\theta-\delta} \\ b \text{ squarefree}}} \left(\frac{x^{\theta}}{2b^{3/2}} \right) + O\left(x^{\theta-\delta}\right).
\end{aligned}$$

It is easy to see that

$$\sum_{b \text{ squarefree}} \frac{1}{b^{3/2}} = \prod_p \left(1 + \frac{1}{p^{3/2}}\right) = \prod_p \left(1 - \frac{1}{p^3}\right) \prod_p \left(1 - \frac{1}{p^{3/2}}\right)^{-1} = \frac{\zeta(3/2)}{\zeta(3)}.$$

One easily deduces that the number of squarefull numbers of the form $a^2b^3 \in (x, x+h]$ with $b \le x^{\theta-\delta}$ is $\zeta(3/2)x^\theta/(2\zeta(3)) + O\left(x^{\theta-\delta}\right)$. Hence, the number of squarefull numbers of the form $a^2b^3 \in (x, x+h]$ with $b \le 2x^{1/3}w^{-2/3}$ is

$$\frac{\zeta(3/2)}{2\zeta(3)}x^\theta + O(S_2) + O\left(x^{\theta-\delta}\right),$$

and the lemma follows. ■

We now estimate S_1 and S_2 with an appropriate choice for w. We take $w = x^{3/13}$. By subdividing the interval $(x, x + x^{(1/2)+\theta}]$ into smaller subintervals if necessary, it suffices to consider $\theta = (5/39) + \epsilon$ where ϵ is a sufficiently small positive constant. To obtain our result, we consider $\delta < \epsilon$ and show that

$$S_1(w) \ll x^{5/39} \log x \quad \text{and} \quad S_2(w) \ll x^{5/39} \log x.$$

To estimate $S_1(w)$ and $S_2(w)$, we define

$$T_1(N) = \sum_{N<n\le 2N} \left(\left[\sqrt[3]{\frac{x+h}{n^2}}\right] - \left[\sqrt[3]{\frac{x}{n^2}}\right] \right) \quad \text{for } x^{\theta-\delta} \le N \le w/2 \tag{21}$$

and

$$T_2(N) = \sum_{N<n\le 2N} \left(\left[\sqrt{\frac{x+h}{n^3}}\right] - \left[\sqrt{\frac{x}{n^3}}\right] \right) \quad \text{for } x^{\theta-\delta} \le N \le x^{1/3}w^{-2/3}. \tag{22}$$

Observe that $T_1(N)$ is the number of positive integral pairs (a, b) for which $a^2b^3 \in (x, x+h]$ and $N < a \le 2N$, and observe that $T_2(N)$ is the number of such pairs (a, b) for which $a^2b^3 \in (x, x+h]$ and $N < b \le 2N$. Straight forward calculations (cf. [12]) show that in fact for each $a \in (N, 2N]$ with N as in (21), there is at most one b such that $a^2b^3 \in (x, x+h]$; also, for each $b \in (N, 2N]$ with N as in (22), there is at most one a such that $a^2b^3 \in (x, x+h]$. In other words, $T_1(N)$ is simply the number of $a \in (N, 2N]$ for which $a^2b^3 \in (x, x+h]$ holds for some integer b, and $T_2(N)$ is simply the number of $b \in (N, 2N]$ for which $a^2b^3 \in (x, x+h]$ holds for some integer a. If $a^2b^3 \in (x, x+h]$, then

$$\sqrt[3]{\frac{x}{a^2}} < b \le \sqrt[3]{\frac{x+h}{a^2}} = \sqrt[3]{\frac{x}{a^2}} + O\left(x^{\theta-(1/6)}N^{-2/3}\right).$$

Thus,

$$\left\| \sqrt[3]{\frac{x}{a^2}} \right\| \ll x^{\theta-(1/6)}N^{-2/3}. \tag{23}$$

Therefore, $T_1(N)$ is bounded by the number of $a \in (N, 2N]$ satisfying (23). Similarly, $T_2(N)$ is bounded by the number of $b \in (N, 2N]$ satisfying

$$\left\|\sqrt{\frac{x}{b^3}}\right\| \ll x^{\theta} N^{-3/2}. \tag{24}$$

The results of this paper are used only in estimating $T_2(N)$; for $T_1(N)$ we appeal to Theorem 3 of Huxley's [22]. We use Theorem 6 with $r = 3$ and $f(u) = \sqrt{x}/u^{3/2}$ to estimate $T_2(N)$. In Theorem 6, we can take $T = \sqrt{x}N^{-3/2}$. With $w = x^{3/13}$, the value of N in (22) is bounded above by $x^{7/39}$. Also, our lower bound on θ above and our choice of δ imply that $N \geq x^{5/39}$. From (24), we see that we can take the value of δ in Theorem 6 (which is different from our use of δ above) to be $O(x^{\theta}N^{-3/2})$. The conditions of Theorem 6 are now met. From Theorem 6, we get $T_2(N) \ll x^{5/39}$. Observe that by subdividing the sum in the definition of S_2 into sums of the form given by $T_2(N)$, we easily deduce that $S_2(w) \ll x^{5/39} \log x$.

We do not elaborate on Huxley's Theorem 3 from [22], but note it is a result similar in nature to our Theorem 6. Using the notation there, we set $F(x) = (x+1)^{-2/3}$, $L = M = N$, $\delta \asymp x^{\theta-(1/6)}N^{-2/3}$ (so that (23) holds), $T = x^{1/3}N^{-2/3}$, and $\Delta = x^{1/3}N^{-8/3}$. One deduces that $T_1(N) \ll x^{5/39}$ and $S_1(w) \ll x^{5/39} \log x$, completing the proof of Theorem 3.

7. The Distribution of Gaps Between Squarefree Numbers

In this section and the next, we will deal with gaps between k−free numbers. This section will specifically treat the case $k = 2$, but it is convenient first to describe some of the background for general k. Let $s_1, s_2, \ldots$ denote the k−free numbers in ascending order. We will be interested in the problem of determining γ for which

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^{\gamma} \sim B(\gamma, k)x \tag{25}$$

where $B(\gamma, k)$ is a constant depending only on γ and k. It is slightly more convenient to deal with establishing

$$\sum_{x/2 < s_{n+1} \leq x} (s_{n+1} - s_n)^{\gamma} \sim \frac{B(\gamma, k)}{2}x \tag{26}$$

which is a necessary and sufficient condition for (25) to hold. Furthermore, for each k, we will restrict our attention to $\gamma \geq 3$ since $\gamma \in [0, 3]$ can be dealt with as in Hooley [21]. One can show without the use of differences how to handle "small" gaps between k−free numbers. More specifically, the following holds.

Lemma 19. *Let $\gamma \in [3, 3k-2)$, and let*

$$0 < \theta < \min\left\{\frac{2(k-1)}{(2k+1)(\gamma-1)}, \frac{k-1}{(k+1)\gamma-(k+3)}\right\} = \begin{cases} \dfrac{2(k-1)}{(2k+1)(\gamma-1)} & \text{if } \gamma \leq 5 \\[2ex] \dfrac{k-1}{(k+1)\gamma-(k+3)} & \text{if } \gamma > 5. \end{cases}$$

Then

$$\sum_{\substack{x/2<s_{n+1}\le x\\ s_{n+1}-s_n\le x^{\theta}}} (s_{n+1}-s_n)^{\gamma} \sim B'(\gamma,k)x$$

for some constant $B'(\gamma,k)$.

We omit the proof of this lemma. This lemma for $k=2$ is based on Lemma 1 of [7] together with an application of a result of Mirsky [34]; the result by Mirsky aids in establishing the lemma in the same manner as it has been applied to help establish (25) since Erdős [3]. For general k, the proof is essentially the same but requires replacing certain estimates for the squarefree case with the corresponding estimates for the $k-$free case. The estimates for the $k-$free case were obtained by Graham and given in Lemma 3 and Lemma 4 of his paper [15].

The idea now is to deal with larger gaps between $k-$free numbers. Observe that by Theorem 1, every gap between $k-$free numbers in $(x/2,x]$ has length $\le cx^{1/(2k+1)}\log x$ for some constant $c=c(k)$. We will take θ as in Lemma 19 and attempt to show that

$$\sum_{\substack{x/2<s_{n+1}\le x\\ x^{\theta}<s_{n+1}-s_n\le cx^{1/(2k+1)}\log x}} (s_{n+1}-s_n)^{\gamma} = o(x).$$

Thus, (26) will follow from Lemma 19 with $B'(\gamma,k)=B(\gamma,k)/2$.

Fix θ as in Lemma 19, and let T be such that $x^{\theta}\le T\le cx^{1/(2k+1)}\log x$. We will estimate the size of $\sum_{T<t\le 2T} N_t$ where N_t is the number of n for which $x/2<s_{n+1}\le x$ and $s_{n+1}-s_n=t$. We are really interested in sums of the form $\sum_{T<t\le 2T} N_t t^{\gamma}$, so we note that there is an obvious relation between these sums, namely

$$T^{\gamma}\sum_{T<t\le 2T} N_t \le \sum_{T<t\le 2T} N_t t^{\gamma} \le (2T)^{\gamma}\sum_{T<t\le 2T} N_t.$$

To estimate $\sum_{T<t\le 2T} N_t$, we establish a connection between this sum and $S(x,X,T)$ which we define as the number of $4-$tuples (p_1,p_2,ℓ_1,ℓ_2) with p_1 and p_2 primes and ℓ_1 and ℓ_2 positive integers satisfying $\ell_2p_2^k>\ell_1p_1^k$, $X<p_i\le 2X$ for each $i\in\{1,2\}$, and for some $I\subseteq[x/2,x]$ with $|I|\le 2T$, $\ell_ip_i^k\in I$ for each $i\in\{1,2\}$.

Observe that in a given gap of length $t\in(T,2T]$ between $k-$free numbers, the number of integers divisible by some p^k with $p\le(1/10)T\log T$ is

$$\le \sum_{p\le(1/10)T\log T}\left(\frac{t}{p^k}+1\right) < t\left(\sum_{n=2}^{\infty}\frac{1}{n^2}\right)+\pi\left((1/10)T\log T\right) < \frac{3}{4}t.$$

Thus, each gap of size $t\in(T,2T]$ between consecutive $k-$free numbers contains at least $t/4>T/4$ integers which are divisible by some p^k with $p>(T/10)\log T$. To deal with the primes $>(T/10)\log T$, we let r be the greatest integer $\le\log x$ and consider $X_j=2^{j-1}(T/10)\log T$ for $j\in\{1,2,\dots,r\}$. Thus, we are interested in primes p belonging to some interval $(X_j, 2X_j]$. Suppose $I = (s_n, s_{n+1})$ for some n with $x/2 < s_{n+1} \le x$ and with $|I| \in (T, 2T]$. For $j \in \{1, 2, \dots, r\}$, define

$$Y_j = Y_j(I) = \left|\{m \in I : \text{ there exists a prime } p \in (X_j, 2X_j] \text{ such that } p^k | m\}\right|.$$

Observe that

$$Y_1 + Y_2 + \cdots + Y_r \ge T/4.$$

Let $\epsilon > 0$ be fixed (depending on k and θ) but sufficiently small, and note that

$$\sum_{j=1}^{\infty} (1+\epsilon)^{-j} = \frac{1}{\epsilon}.$$

Then there is a $j = j(I) = j(I, \epsilon) \in \{1, \dots, r\}$ such that

$$Y_j \ge \frac{\epsilon}{4}(1+\epsilon)^{-j} T. \tag{27}$$

We view x as being sufficiently large so that in particular $T \ge x^\theta$ implies $Y_j \ge 2$.

We now consider a fixed $j \in \{1, \dots, r\}$ and estimate the number of intervals $I = (s_n, s_{n+1})$ with $x/2 < s_{n+1} \le x$, $|I| \in (T, 2T]$, and $j(I) = j$. By (27), each such I contributes at least $\binom{Y_j}{2} \gg_\epsilon (1+\epsilon)^{-2j} T^2$ different 4−tuples $(p_1, p_2, \ell_1, \ell_2)$ counted by $S(x, X_j, T)$. Thus,

$$\sum_{T < t \le 2T} N_t \ll_\epsilon \sum_{j=1}^{r} \frac{S(x, X_j, T)}{T^2}(1+\epsilon)^{2j}. \tag{28}$$

Hence, we can bound the left-hand side of (28) by bounding $S(x, X, T)$. This leads to our next lemma.

Lemma 20. *For $X \ge (1/10) T \log T$,*

$$S(x, X, T) \ll \frac{xT}{X^{2k-2} \log^2 X} + \frac{TX^2}{\log^2 X}$$

and

$$S(x, X, T) \ll \frac{x}{X^{k-1} \log X}.$$

For the case $k = 2$, the first of these estimates is contained in [7] and the second is contained in [21]. For general k, these estimates can be found in [15]. We note that these estimates are necessary in the approach we discussed for establishing Lemma 19.

In this section and the next, we will make use of two different ideas depending on whether $k = 2$ or $k \ge 3$. We are ready now to restrict ourselves in this section to the case $k = 2$. We will follow the strategy given in [7] which makes use of differences to show that for intervals I with $|I| \ge x^\theta$ as above, $j(I)$ must be large. There are difficulties that arise in trying a direct generalization of this idea for general k to make an improvement on the work of Graham in [15]. On the other hand, it is worth noting here that in the next section, the new idea we will present alleviates these difficulties. Nevertheless, we will not pursue generalizing here the ideas in this section to $k-$free numbers as the authors feel that any resulting improvement would be minor.

Let $k=2$. To prove Theorem 4, we fix $\gamma < 43/13$. Then in Lemma 19, we can take $\theta > 13/75$. Fixing θ as such, we consider an interval $I=(s_n, s_{n+1})$ with $|I| = t \in (T, 2T]$ where $T \geq x^{\theta}$. Let $X \geq (T/10)\log T$. Define

$$Y(I) = \left|\{m \in I : \text{ there exists a prime } p \in (X, 2X] \text{ such that } p^2 | m\}\right|.$$

Since $X \geq (T/10)\log T$, we have that for each $p \in (X, 2X]$, there is at most one $m \in I$ such that $p^2|m$. It is clear then that $Y(I) \leq |S(I)|$ where

$$S(I) = \{u \in (X, 2X] \cap \mathbb{Z} : \text{ there exists an integer } m \in I \text{ such that } u^2|m\}.$$

Let $\eta > 0$ be sufficiently small and independent of ϵ. Our immediate goal is to show that

$$|S(I)| \ll T^{1-\eta} \quad \text{for } x^{\theta} \leq T \leq x^{0.22} \text{ and } (T/10)\log T \leq X \leq T^{15/13} \tag{29}$$

so that the corresponding bound on $Y(I)$ holds. Fix $y \in I \subseteq (x/4, x]$, and observe that if $u \in S(I)$, then

$$\left|\left|\frac{y}{u^2}\right|\right| < \frac{|I|}{u^2} \leq 2TX^{-2}.$$

We are ready to apply Theorem 6; however, we note that T has another meaning in the statement of Theorem 6, so for convenience here we will refer to T in Theorem 6 as T' (and fix the use of T as in the discussion above). We take $f(u) = y/u^2$ and $\delta = 2TX^{-2}$. Here, $N = X$ and $T' = yX^{-2}$. We consider two cases in (29) depending on whether $T \geq x^{0.186}$ or $T < x^{0.186}$. In the first case, one easily checks that the bound in (29) follows from Theorem 6 with $r=4$; in the second case, the bound in (29) follows from Theorem 6 with $r=5$. Hence, (29) holds.

From (29), we have that $Y(I) \ll T^{1-\eta}$ if $x^{\theta} \leq T \leq x^{0.22}$ and $(T/10)\log T \leq X \leq T^{15/13}$. Fix T satisfying $x^{\theta} \leq T \leq x^{0.22}$. Then with $j = j(I)$, we must have $X_j > T^{15/13}$; otherwise, $Y_j(I) \ll T^{1-\eta}$ would hold, which contradicts (27). Thus,

$$X_j > T^{15/13} \quad \text{whenever } |I| \in (T, 2T] \text{ where } x^{\theta} \leq T \leq x^{0.22}.$$

The argument which gave (28) can now be adjusted to give that (28) holds with the sum restricted to those j for which $X_j > T^{15/13}$. We are now ready to use (28) so modified and Lemma 20. We consider three possibilities in (28) for the size of X_j. For $X_j \leq x^{1/4}$, we observe that Lemma 20 with $k=2$ implies

$$\frac{S(x, X_j, T)}{T^2} \ll \frac{x}{TX_j^2 \log^2 X_j} \ll \frac{x}{T^{43/13}\log T}.$$

Recall that with $k = 2$ we are only interested in $T \le cx^{1/5} \log x$ where c is as in Theorem 1. For $x^{1/4} < X_j \le x^{4/15}$, we obtain from Lemma 20 that

$$\frac{S(x, X_j, T)}{T^2} \ll \frac{X_j^2}{T \log^2 X_j} \ll \frac{x^{8/15}}{T \log T} \ll \frac{x}{T^{43/13} \log T}.$$

For $X_j > x^{4/15}$, we have

$$X_j > (x^{1/5})^{4/3} > T^{1.33}$$

so that from Lemma 20

$$\frac{S(x, X_j, T)}{T^2} \ll \frac{x}{T^2 X_j \log X_j} \ll \frac{x}{T^{3.33} \log T} \ll \frac{x}{T^{43/13} \log T}.$$

Therefore, with $x^\theta \le T \le cx^{1/5} \log x$,

$$\sum_{T < t \le 2T} N_t t^\gamma \ll_\epsilon T^\gamma \sum_{\substack{1 \le j \le r \\ X_j > T^{15/13}}} \frac{S(x, X_j, T)}{T^2} (1+\epsilon)^{2j} \ll_\epsilon xT^{\gamma - (43/13)} (1+\epsilon)^{2 \log x}.$$

By breaking up the interval $(x^\theta, cx^{1/5} \log x]$ into subintervals of the form $(T, 2T]$ and summing, we obtain

$$\begin{aligned} \sum_{\substack{x/2 < s_{n+1} \le x \\ x^\theta < s_{n+1} - s_n \le cx^{1/5} \log x}} (s_{n+1} - s_n)^\gamma &= \sum_{x^\theta < t \le cx^{1/5} \log x} N_t t^\gamma \\ &\ll_\epsilon x^{1 + \theta(\gamma - (43/13))} (1+\epsilon)^{2 \log x}. \end{aligned}$$

Since $\gamma < 43/13$ and ϵ is sufficiently small, we deduce that

$$\sum_{\substack{x/2 < s_{n+1} \le x \\ x^\theta < s_{n+1} - s_n \le cx^{1/5} \log x}} (s_{n+1} - s_n)^\gamma = o(x),$$

which implies Theorem 4.

8. The Distribution of Gaps Between k-Free Numbers

Let $s_1, s_2, \ldots$ denote the k-free numbers in ascending order with $k \ge 2$. In the previous section, we began our efforts to establish (26) and, hence, (25) for $3 \le \gamma < 2k - 1$. Lemma 19 enables us to deal with small gaps between k-free numbers. It is worth noting the strength of Lemma 19. By Theorem 1, we know that $s_{n+1} - s_n \le cx^{1/(2k+1)} \log x$ for all n with $x/2 < s_{n+1} \le x$. By considering $\theta > 1/(2k+1)$, Lemma 19 allows us to deal with all such gaps whenever $k \ge 3$ and $\gamma < 2k - 2 + (4/(k+1))$. In other words, the result of Graham [15] follows from Lemma 19. The use of differences for Graham's result only occurs indirectly in the use of Theorem 1. Here, we shall likewise only make use of differences by using Theorem 1.

As in the previous section, we set $r = [\log x]$. For $i \in \{1, 2, \dots, r\}$, we define

$$N_t(i) = |\{I = (s_n, s_{n+1}) : x/2 < s_{n+1} \le x, |I| = t, j(I) = i\}|$$

and

$$N = N(T, i) = |\{I = (s_n, s_{n+1}) : x/2 < s_{n+1} \le x, |I| \in (T, 2T], j(I) = i\}|$$

so that

$$N = \sum_{T < t \le 2T} N_t(i).$$

Recall the notation N_t in our previous section. Thus, $N_t = \sum_{i=1}^{r} N_t(i)$. In (28), we found a bound for $\sum_{T<t\le 2T} N_t$ in terms of the expressions $S(x, X_j, T)$. It follows easily from our reasoning there that

$$N \ll \frac{S(x, X_i, T)}{T^2}(1+\epsilon)^{2i}, \tag{30}$$

where here and throughout this section implied constants may depend on ϵ (and, hence, θ) as well as k. We will want to make use of this bound here, but we will also want a new bound in terms of $S'(x, X_i, T)$ which we define as the number of quadruples $(p_1, p_2, \ell_1, \ell_2)$ for which p_1 and p_2 are primes, $X_i < p_1 < p_2 \le 2X_i$, $p_1^k \ell_1$ and $p_2^k \ell_2$ are both in some interval $I = (s_n, s_{n+1})$ with $x/2 < s_{n+1} \le x$, $|I| \in (T, 2T]$, $j(I) = i$, and if a prime $p > p_1$ is such that a multiple of p^k is in I, then $p \ge p_2$. This latter condition is simply asserting that if $(p_1, p_2, \ell_1, \ell_2) \in S'(x, X_i, T)$, then p_1 and p_2 are consecutive primes in the set of primes p for which p^k divides some element of I. Recall that Lemma 19 enables us to restrict our attention to $T \ge x^\theta$ for some $\theta > 0$ so that, in particular, $Y_{j(I)} \ge 2$ with I as in the definition of $S'(x, X_i, T)$. Also, for $I = (s_n, s_{n+1})$ with $|I| \le 2T$ and for $p \in (X, 2X]$ with $X \ge (T/10)\log T$, we can have at most one multiple of p^k in I.

Lemma 21. *Let $T \ge x^\theta$ where θ is as in Lemma 19. For $i \in \{1, 2, \dots, r\}$,*

$$N(T, i) \ll \frac{S'(x, X_i, T)}{T}(1+\epsilon)^i.$$

Proof. Let $I = (s_n, s_{n+1})$ be an interval with $x/2 < s_{n+1} \le x$, $|I| = t \in (T, 2T]$, and $j(I) = i$. Then by the definition of $j(I)$, $Y_i(I) \ge \epsilon(1+\epsilon)^{-i}T/4$. By the definition of Y_i, there are Y_i elements of I of the form $p^k\ell$ where ℓ is a positive integer and p is a prime $\in (X_i, 2X_i]$. By the definition of $S'(x, X_i, T)$, each such interval I contributes $\gg (1+\epsilon)^{-i}T$ of the quadruples $(p_1, p_2, \ell_1, \ell_2)$ counted by $S'(x, X_i, T)$. The result easily follows. ■

For each of the intervals I counted by $N(T, i)$, there is an ordering of the quadruples $(p_1, p_2, \ell_1, \ell_2)$ counted by $S'(x, X_i, T)$ with $p_1^k \ell_1 \in I$ corresponding to the size of p_1. We fix $I = (s_n, s_{n+1})$ as in the definition of $N(T,i)$. With the ordering just described, we suppose now that we have consecutive elements

$$(p, p+a_1, \ell_0, \ell_1), (p+a_1, p+a_1+a_2, \ell_1, \ell_2), \dots,$$
$$(p+a_1+\cdots+a_{2k-1}, p+a_1+\cdots+a_{2k}, \ell_{2k-1}, \ell_{2k})$$

where

$$(p+a_1+\cdots+a_j)^k \ell_j \in I \quad \text{for } j \in \{0,1,\dots,2k\}.$$

Let

$$d(j,j') = \gcd(\ell_j, \ell_{j'}) \quad \text{for } 0 \le j < j' \le 2k.$$

Let A be a real number ≥ 1. We claim that either

(i) $a_1 + \cdots + a_{2k} > A$ or

(ii) $\min_{0\le j<j'\le 2k}\{d(j,j')\} \le 2xA^{2k-1}X_i^{-2k}$.

To establish the claim, we assume (i) and (ii) do not hold and make use of some aspects of the proof of Theorem 7. In particular, we note that these specific aspects of the proof of Theorem 7 are based on ideas from Halberstam and Roth [18] and Nair [36]. Halberstam and Roth established the existence of homogeneous polynomials $P(u,a)$ and $Q(u,a)$ of degree $k-1$ in the variables u and a such that $P(0,a) \not\equiv 0$, $Q(0,a) \not\equiv 0$, and

$$P(u,a)(u+a)^k - Q(u,a)u^k = a^{2k-1}.$$

We will use this identity, but the fact that the right-hand side is a non-zero multiple of a^{2k-1} will be sufficient for our purposes and the existence of such polynomials follows from Lemma 8 with $\ell = k-1$ and $s = k$ (the left-hand side of (12) is a polynomial in u and a which forces all but one of the terms on the right-hand side of (12) to be 0). Note that $P(u,a) \not\equiv Q(u,a)$ easily follows from the identity above.

Let $y = s_n$ and consider integers j and j' such that $0 \le j < j' \le 2k$. Then

$$(p+a_1+\cdots+a_j)^k \frac{\ell_j}{d(j,j')} \quad \text{and} \quad (p+a_1+\cdots+a_{j'})^k \frac{\ell_{j'}}{d(j,j')} \quad \text{are in} \quad \left(\frac{y}{d(j,j')}, \frac{y+2T}{d(j,j')}\right].$$

Thus,

$$\frac{\ell_j}{d(j,j')} = \frac{y}{d(j,j')(p+a_1+\cdots+a_j)^k} + O\left(\frac{T}{d(j,j')X_i^k}\right)$$

and

$$\frac{\ell_{j'}}{d(j,j')} = \frac{y}{d(j,j')(p+a_1+\cdots+a_{j'})^k} + O\left(\frac{T}{d(j,j')X_i^k}\right).$$

Hence,

$$P((p+a_1+\cdots+a_j),(a_{j+1}+\cdots+a_{j'}))\frac{\ell_j}{d(j,j')}$$

$$-\,Q\left((p+a_1+\cdots+a_j),(a_{j+1}+\cdots+a_{j'})\right)\frac{\ell_{j'}}{d(j,j')}$$

$$=\frac{(a_{j+1}+\cdots+a_{j'})^{2k-1}y}{d(j,j')(p+a_1+\cdots+a_j)^k(p+a_1+\cdots+a_{j'})^k}+O\left(\frac{T}{d(j,j')X_i}\right).$$

Observe that $y\le x$ and each of $p+a_1+\cdots+a_j$ and $p+a_1+\cdots+a_{j'}$ is $\ge X_i$. Also, since we are assuming (i) and (ii) do not hold, we deduce that $a_{j+1}+\cdots+a_{j'}\le A$ and $d(j,j')>2xA^{2k-1}X_i^{-2k}$. These imply that the first term on the right-hand side above is $<1/2$. Clearly, it is also positive. Since $X_i\ge(T/10)\log T$ and $T\ge x^\theta$ with x sufficiently large and $\theta>0$, we also deduce that the error term on the right-hand side above has absolute value $<1/2$. Since the left-hand side is an integer, we obtain that

$$P((p+a_1+\cdots+a_j),(a_{j+1}+\cdots+a_{j'}))\ell_j$$

$$-\,Q\left((p+a_1+\cdots+a_j),(a_{j+1}+\cdots+a_{j'})\right)\ell_{j'}=0.$$

This equation holds for every choice of j and j' with $0\le j<j'\le 2k$. With $j=0$ and $j'=1$, the equation is analogous to the situation we had in (16). A precisely analogous argument to what followed in the remainder of that paragraph leads us to deduce that we cannot have the $2k+1$ numbers $p,p+a_1,\ldots,p+a_1+\cdots+a_{2k}$ above; in other words, we obtain a contradiction to the assumption that (i) and (ii) do not hold. Thus, either (i) or (ii) holds.

For each of the N intervals $I=(s_n,s_{n+1})$ defined by $N(T,i)$, we consider the consecutive elements counted by $S'(x,X_i,T)$. By disregarding $\le 2k-1$ of the largest of these consecutive elements (in terms of the size of the first component) we can express the remaining consecutive elements as a union of disjoint subsets with each subset consisting of exactly $2k$ consecutive elements. These $2k$ consecutive elements counted by $S'(x,X_i,T)$ correspond to $2k+1$ numbers $(p+a_1+\cdots+a_j)^k\ell_j$ for $0\le j\le 2k$ as above. Since either (i) or (ii) holds, we can find $p_1^k\ell_1$ and $p_2^k\ell_2$ from among these such that either $p_2-p_1>A$ or $\gcd(\ell_1,\ell_2)\le 2xA^{2k-1}X_i^{-2k}$. As I varies, we consider the subsets as described above of size $2k$ and create new quadruples (p_1,p_2,ℓ_1,ℓ_2), one quadruple for each subset. The set of all new quadruples, we call W_i. The set W_i has the property that

$$S'(x,X_i,T)\le 2k(|W_i|+N) \tag{31}$$

and for each element (p_1,p_2,ℓ_1,ℓ_2) of W_i, either $p_2-p_1>A$ or $\gcd(\ell_1,\ell_2)\le 2xA^{2k-1}X_i^{-2k}$. Since $T\ge x^\theta$ and ϵ is sufficiently small, we can deduce from Lemma 21 that $N\le c_9S'(x,X_i,T)$ where $c_9>0$ is as small as we wish ($c_9=1/(6k)$ will do). From (31), we deduce that

$$S'(x,X_i,T)\le 3k|W_i|. \tag{32}$$

Lemma 22. *If $X_i^{2k} > 4Tx$ and $T \geq x^\theta$ where θ is as in Lemma 19, then $S'(x, X_i, T) \ll xX_i(1+\epsilon)^{2ki}/T^{2k}$.*

Proof. Let $A = (X_i/T)(1+\epsilon)^i(\log x)^{1/(2k)}$. For each of the N gaps between $k-$free numbers counted by $N(T, i)$, it follows from the definition of W_i that there are at most $2X_i/A$ numbers $p_1^k\ell_1$ and $p_2^k\ell_2$ with $(p_1, p_2, \ell_1, \ell_2)$ in W_i and $p_2 - p_1 > A$. Thus, we have the upper bound $2X_iN/A$ on the number of elements $(p_1, p_2, \ell_1, \ell_2)$ in W_i with $p_2 - p_1 > A$.

Let $(p_1, p_2, \ell_1, \ell_2)$ and $(p_1, p_2, \ell_1', \ell_2')$ be elements of W_i. We show that $\ell_1/\ell_2 = \ell_1'/\ell_2'$. Assume otherwise. Observe that each of ℓ_1, ℓ_1', ℓ_2, and ℓ_2' is $\leq x/X_i^k$. Thus,

$$\frac{1}{\ell_2\ell_2'} \leq \left|\frac{\ell_1}{\ell_2} - \frac{\ell_1'}{\ell_2'}\right| \leq \left|\frac{\ell_1}{\ell_2} - \frac{p_2^k}{p_1^k}\right| + \left|\frac{\ell_1'}{\ell_2'} - \frac{p_2^k}{p_1^k}\right| \leq \frac{2T}{\ell_2 p_1^k} + \frac{2T}{\ell_2' p_1^k},$$

implies

$$X_i^k \leq p_1^k \leq 2T(\ell_2' + \ell_2) \leq \frac{4Tx}{X_i^k}.$$

This contradicts the first condition in the lemma. Hence, $\ell_1/\ell_2 = \ell_1'/\ell_2'$.

Let d be a positive integer, and suppose that $(p_1, p_2, \ell_1, \ell_2)$ and $(p_1, p_2, \ell_1', \ell_2')$ are elements of W_i with $\gcd(\ell_1, \ell_2) = \gcd(\ell_1', \ell_2') = d$. Then we can deduce from the above that $\ell_1/\ell_2 = \ell_1'/\ell_2'$ from which we easily obtain $\ell_1 = \ell_1'$ and $\ell_2 = \ell_2'$.

For each positive integer d, we consider the quadruples $(p_1, p_2, \ell_1, \ell_2)$ in W_i such that $\gcd(\ell_1, \ell_2) = d$ and such that if $(p_1, p_2, \ell_1', \ell_2')$ is in W_i, then $\gcd(\ell_1', \ell_2') \geq d$. Call this set $T(d)$. For a given pair of primes (p_1, p_2), we have just shown that there is at most one quadruple $(p_1, p_2, \ell_1, \ell_2)$ in $T(d)$. The definition of $T(d)$ therefore implies that for each pair of primes (p_1, p_2), there is either zero or exactly one choice of positive integers d, ℓ_1, and ℓ_2 for which $(p_1, p_2, \ell_1, \ell_2)$ is in $T(d)$.

Define $T(d, a)$ as the set of $(p_1, p_2, \ell_1, \ell_2)$ in $T(d)$ for which $p_2 - p_1 = a$. Suppose $(p_1, p_2, \ell_1, \ell_2)$ is in $T(d, a)$. Suppose further that $(p_1, p_2, \ell_1', \ell_2')$ is in W_i and $\gcd(\ell_1', \ell_2') = d' > d$. We show that $d' < 3d$. Write $\ell_1 = dm_1$, $\ell_2 = dm_2$, $\ell_1' = d'm_1'$, and $\ell_2' = d'm_2'$. As we have already seen, $\ell_1/\ell_2 = \ell_1'/\ell_2'$ so that $m_1 = m_1'$ and $m_2 = m_2'$. Since both $p_1^k dm_1$ and $p_1^k d'm_1$ are in $(x/3, x]$, we get that $d' < 3d$ as desired. This implies that the number of elements $(p_1, p_2, \ell_1, \ell_2)$ in W_i with $p_2 - p_1 \leq A$ is bounded above by $\sum_{a \leq A}\sum_{d=1}^{\infty} 2d|T(d, a)|$. From the definition of A, we deduce that

$$|W_i| \leq \sum_{a \leq A}\sum_{d=1}^{\infty}(2d|T(d, a)|) + E$$

where

$$E \leq \frac{2X_iN}{A} \ll \frac{TN}{(1+\epsilon)^i(\log x)^{1/(2k)}}.$$

Using (32) and the bound on N given by Lemma 21, we obtain $E \leq |W_i|/3$. Therefore,

$$|W_i| \ll \sum_{a \leq A}\sum_{d=1}^{\infty} d|T(d, a)|.$$

Note that $|T(d,a)|$ counts the number of primes p such that $(p, p+a, \ell_1, \ell_2)$ is in $T(d)$ for some integers ℓ_1 and ℓ_2. Since the pair $(p, p+a)$ corresponds to either zero or exactly one choice of positive integers d, ℓ_1, and ℓ_2 such that $(p, p+a, \ell_1, \ell_2)$ is in $T(d)$, we deduce from the Prime Number Theorem that for each positive integer a,

$$\sum_{d=1}^{\infty} |T(d,a)| \ll \frac{X_i}{\log X_i}.$$

Recall that for each element $(p_1, p_2, \ell_1, \ell_2)$ of W_i, either $p_2 - p_1 > A$ or $\gcd(\ell_1, \ell_2) \leq 2xA^{2k-1}X_i^{-2k}$. Thus, if $(p, p+a, \ell_1, \ell_2)$ is in $T(d)$ and $a \leq A$, then $d = \gcd(\ell_1, \ell_2) \leq 2xA^{2k-1}X_i^{-2k}$. Therefore,

$$\begin{aligned}
\sum_{a \leq A} \sum_{d=1}^{\infty} d|T(d,a)| &= \sum_{a \leq A} \sum_{d \leq 2xA^{2k-1}X_i^{-2k}} d|T(d,a)| \\
&\leq 2xA^{2k-1}X_i^{-2k} \sum_{a \leq A} \sum_{d \leq 2xA^{2k-1}X_i^{-2k}} |T(d,a)| \\
&\leq 2xA^{2k-1}X_i^{-2k} \sum_{a \leq A} \sum_{d=1}^{\infty} |T(d,a)| \ll xA^{2k}X_i^{-2k+1}/\log X_i.
\end{aligned}$$

Using that $X_i \geq x^{1/(2k)}$ from the conditions in the lemma and using our choice of A, the lemma follows. ■

Theorem 5 follows from Theorem 4 in the case $k = 2$. Let $k \geq 3$, and let θ be as in Lemma 19. We consider several estimates for $N(T,i)$ depending on the relative size of X_i with respect to x and T. Recall that $X_i \geq (T/10)\log T$. If $X_i \leq x^{1/(2k)}$, we obtain from Lemma 20 and (30) that

$$N(T,i) \ll \frac{x}{X_i^{2k-2} T \log^2 X_i}(1+\epsilon)^{2i} \ll \frac{x}{T^{2k-1}}(1+\epsilon)^{2i}.$$

If $x^{1/(2k)} < X_i \leq (4Tx)^{1/(2k)}$, we obtain from Lemma 20 and (30) that

$$N(T,i) \ll \frac{X_i^2}{T \log^2 X_i}(1+\epsilon)^{2i}.$$

Using that $k \geq 3$ and $T \leq cx^{1/(2k+1)}\log x$, one easily checks that this last asymptotic inequality implies

$$N(T,i) \ll \frac{x}{T^{2k-1}}(1+\epsilon)^{2i}.$$

For $(4Tx)^{1/(2k)} < X_i \leq T^{(2k-1)/k}$, we use Lemma 21 and Lemma 22 to obtain

$$N(T,i) \ll \frac{xX_i}{T^{2k+1}}(1+\epsilon)^{(2k+1)i} \ll xT^{-(2k^2-k+1)/k}(1+\epsilon)^{(2k+1)i}.$$

Finally, for $X_i > T^{(2k-1)/k}$, Lemma 20 and (30) imply that

$$N(T,i) \ll \frac{x}{X_i^{k-1}T^2 \log X_i}(1+\epsilon)^{2i} \ll xT^{-(2k^2-k+1)/k}(1+\epsilon)^{2i}.$$

We use that

$$\sum_{T<t\leq 2T} N_t = \sum_{T<t\leq 2T} \sum_{i=1}^{r} N_t(i) = \sum_{i=1}^{r} N(T,i).$$

Since $r \leq \log x$, we can combine the above estimates to obtain

$$\sum_{T<t\leq 2T} N_t \ll \frac{x}{T^{2k-1}}(1+\epsilon)^{(2k+1)\log x} \log x$$

so that

$$\sum_{T<t\leq 2T} N_t t^{\gamma} \ll xT^{\gamma-(2k-1)}(1+\epsilon)^{(2k+1)\log x} \log x.$$

For ϵ sufficiently small and $\gamma < 2k-1$, we get

$$\sum_{\substack{x/2<s_{n+1}\leq x \\ x^{\theta}<s_{n+1}-s_n\leq cx^{1/(2k+1)}\log x}} (s_{n+1}-s_n)^{\gamma} = o(x),$$

from which (26) and, hence, Theorem 5 follows.

ACKNOWLEDGMENTS: The authors are very grateful to M.N. Huxley for comments which helped to improve on this paper. In particular, the authors followed his suggestions on the demonstration of Theorem 6 which led to the removal of an additional error term that existed in a previous version of this paper.

References

1. G.A. Baker and P. Graves-Morris, *Padé Approximants, Part I: Basic Theory (Encyclopedia of Mathematics, Volume 13)*, Addison-Wesley Publ. Co., Reading, MA, 1981.
2. P.T. Bateman and E. Grosswald, *On a theorem of Erdős and Szekeres*, Illinois J. Math. **2** (1958), 88–98.
3. P. Erdős, *Some problems and results in elementary number theory*, Publ. Math. Debrecen **2** (1951), 103–109.
4. P. Erdős and A. Ivić, *The distribution of values of a certain class of arithmetic functions at consecutive integers*, Papers from the conference held in Budapest, July 20–25, 1987, edited by K. Győry and G. Halász, *Colloquia Mathematica Societatis János Bólyai*, 51, North-Holland Publishing Co., Amsterdam, 1990, 45–91.

5. M. Filaseta, *An elementary approach to short intervals results for k–free numbers*, J. Number Theory **30** (1988), 208–225.
6. M. Filaseta, *Short interval results for squarefree numbers*, J. Number Theory **35** (1990), 128–149.
7. M. Filaseta, *On the distribution of gaps between squarefree numbers*, Mathematika **40** (1993), 88–101.
8. M. Filaseta, *Short interval results for k-free values of irreducible polynomials*, Acta Arith. **64** (1993), 249–270.
9. M. Filaseta, *Powerfree values of binary forms*, Journal of Number Theory **49** (1994), 250–268.
10. M. Filaseta and O. Trifonov, *On gaps between squarefree numbers*, Analytic Number Theory, Proceedings of a Conference in Honor of Paul T. Bateman (Progress in Mathematics Series, Vol. 85), edited by Berndt, Diamond, Halberstam, and Hildebrand, Birkhäuser, Boston (1990), 235–253.
11. M. Filaseta and O. Trifonov, *On gaps between squarefree numbers II*, J. London Math. Soc. (2) **45** (1992), 215–221.
12. M. Filaseta and O. Trifonov, *The distribution of squarefull numbers in short intervals*, Acta Arith. **67** (1994), 323–333.
13. E. Fogels, *On the average values of arithmetic functions*, Proc. Cambridge Philos. Soc. **37** (1941), 358–372.
14. E. Fouvry and H. Iwaniec, *Exponential sums with monomials*, J. Number Theory **33** (1989), 311–333.
15. S. W. Graham, *Moments of gaps between $k-$free numbers*, Journal of Number Theory **44** (1993), 105–117.
16. S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums, London Math. Soc. Lecture Note Series 126*, Cambridge Univ. Press, Cambridge, 1991.
17. S. W. Graham and G. Kolesnik, *On the difference between consecutive squarefree integers*, Acta Arith. **49** (1988), 435–447.
18. H. Halberstam and K. F. Roth, *On the gaps between consecutive k–free integers*, J. London Math. Soc. (2) **26** (1951), 268–273.
19. D.R. Heath-Brown, *Square-full numbers in short intervals*, Math. Proc. Camb. Phil. Soc. **110,1** (1991), 1–3.
20. I.N. Herstein, *Topics in Algebra (2nd edition)*, Xerox College Publishing, Toronto, 1975.
21. C. Hooley, *On the distribution of square-free numbers*, Can. J. Math. **25** (1973), 1216-1223.
22. M.N. Huxley, *The integer points close to a curve*, Mathematika **36** (1989), 198–215.
23. M. N. Huxley and M. Nair, *Power free values of polynomials III*, Proc. London Math. Soc. **41** (1980), 66–82.
24. M. N. Huxley and P. Sargos, *Points entiers au voisinage d'une courbe plane de classe* $\mathbb{C}^n$, Acta Arith. **69** (1995), 359–366.
25. E. Isaacson and H.B. Keller, *Analysis of Numerical Methods*, John Wiley & Sons, New York, 1966.

26. A. Ivić, *On the number of finite non-isomorphic abelian groups in short intervals*, Math. Nachr. **101** (1981), 257–271.
27. C.-H. Jia, *The distribution of squarefull numbers*, Chinese, Acta Scientiarum Naturalium, Univ. Pekinensis **3** (1987), 21–27.
28. C.-H. Jia, *On squarefull integers in short intervals*, Chinese, Acta Math. Sinica **5** (1987), 614–621.
29. E. Krätzel, *Die Werteverteilung der Anzahl der nicht-isomorphen Abelschen Gruppen endlicher Ordnung in kurzen Intervallen*, Math. Nachr. **98** (1980), 135–144.
30. E. Krätzel, *Lattice Points*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1988.
31. H. Li, *On the number of finite non-isomorphic abelian groups in short intervals*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 1–5.
32. H. Liu, *On square-full numbers in short intervals*, Chinese, Acta Math. Sinica **6** (1990), 148–164.
33. H. Liu, *The number of squarefull numbers in an interval*, Acta Arith. **64** (1993), 129–149.
34. L. Mirsky, *Arithmetical pattern problems related to divisibility by r-th powers*, Proc. London Math. Soc. **50** (1949), 497–508.
35. M. Nair, *Power free values of polynomials*, Mathematika **23** (1976), 159–183.
36. M. Nair, *Power free values of polynomials II*, Proc. London Math. Soc. (3) **38** (1979), 353–368.
37. R. A. Rankin, *Van der Corput's method and the theory of exponent pairs*, Quart. J. Math. Oxford Ser. (2) **6** (1955), 147–153.
38. H. E. Richert, *On the difference between consecutive squarefree numbers*, J. London Math. Soc. (2) **29** (1954), 16–20.
39. K. F. Roth, *On the gaps between squarefree numbers*, J. London Math. Soc. (2) **26** (1951), 263–268.
40. P. G. Schmidt, *Abschätzungen bei unsymmetrischen Gitterpunktproblemen*, Dissertation zur Erlangung des Doktorgrades der Mathematisch–Naturwissenschaftlichen Fakultät der Georg–AugustUniversität zu Göttingen, 1964.
41. P. G. Schmidt, *Zur Anzahl quadratvoller Zahlen in kurzen Intervallen und ein verwandtes Gitterpunktproblem*, Acta Arith. **50** (1988), 195–201.
42. P. G. Schmidt, *Über dier Anzahl quadratvoller Zahlen in kurzen Intervallen*, Acta Arith. **46** (1986), 159–164.
43. A. Schinzel and E. Wirsing, *Multiplicative properties of the partition function*, Proc. Indian Acad. Sci. (Math. Sci.) **97**, Nos. 1–3 (1987), 297–303.
44. P. Shiu, *On the number of square-full integers between successive squares*, Mathematika **27** (1980), 171–178.
45. P. Shiu, *On squarefull integers in a short interval*, Glasgow Math. J. **25** (1984), 127–134.
46. O. Trifonov, *On the squarefree problem*, C. R. Acad. Bulgare Sci. **41** (1988), 37–40.
47. O. Trifonov, *On the squarefree problem II*, Mathematica Balcanika **3** (1989), 284–295.
48. O. Trifonov, *On the gaps between consecutive k-free numbers*, Mathematica Balcanika **4** (1990), 50–60.
49. O. Trifonov, *On gaps between k−free numbers*, J. of Number Theory **55** (1955), 46–59.