AN ESTIMATE FOR THE NUMBER OF REDUCIBLE BESSEL POLYNOMIALS OF BOUNDED DEGREE

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1. Introduction

The nth Bessel polynomial is

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{2^j (n-j)! j!} x^j.$$

In [3], E. Grosswald conjectured that $y_n(x)$ is irreducible over the rationals for every positive integer n. In [1], the first author proved that almost all $y_n(x)$ are irreducible and later [2] sharpened this by showing that the number of $n \le t$ for which $y_n(x)$ is reducible is $\ll t/\log\log\log t$. The object of this paper is to give a further sharpening.

Theorem. *The number of $n \le t$ for which $y_n(x)$ is reducible is $\ll t^{2/3}$.*

The first author's earlier work used the Tchebotarev Density Theorem, but the proof given here uses only elementary estimates. Our starting point is the Corollary to Lemma 2 in [1], which states that if

$$(1) \qquad \Bigg(\prod_{p \mid n(n+1)} p\Bigg)^2 \Bigg(\prod_{\substack{p \mid (n-1) \\ p \text{ odd}}} p\Bigg) \Bigg(\prod_{\substack{p \mid (n+2) \\ p > 3}} p\Bigg) > n^2(n+1)^2,$$

then $y_n(x)$ is irreducible. We shall show that (1) holds for most n by showing that the non-squarefree part of $(n-1)n(n+1)(n+2)$ is typically very small.

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2. Preliminaries

For every positive integer n, we define

$$a_n = \prod_{\substack{p^\alpha \| n \\ \alpha \text{ odd}}} p \text{ and } b_n = \prod_{p^\alpha \| n} p^{[\alpha/2]},$$

where $p^\alpha \| n$ denotes, as usual, that p^α is the highest power of p dividing n. We then have that $n = a_n b_n^2$ and that

$$a_n \le \prod_{p|n} p. \tag{2}$$

In the next lemma, we use (2) to state (1) in a more usable form.

Lemma 1. *If $y_n(x)$ is reducible and $t < n \le 2t$ then*

$$b_{n-1} b_n^2 b_{n+1}^2 b_{n+2} > \tfrac{1}{3} t.$$

Proof. From (1) and (2), we see that if $y_n(x)$ is reducible, then

$$\frac{n-1}{b_{n-1}^2} \cdot \frac{n^2}{b_n^4} \cdot \frac{(n+1)^2}{b_{n+1}^4} \cdot \frac{n+2}{b_{n+2}^2} \le 6n^2(n+1)^2.$$

The result now follows.

Lemma 2. *If y is a positive real number, then*

$$\#\{n \in (t, 2t] : b_n > y\} \ll \frac{t}{y} + t^{1/2}.$$

Proof. The left–hand side is at most

$$\sum_{t<n\le 2t} \sum_{\substack{b^2 | n \\ b>y}} 1 \ll \sum_{y<b\le\sqrt{2t}} \left(\frac{t}{b^2} + 1\right) \ll \frac{t}{y} + t^{1/2}.$$

Lemma 3. *If $z \geq 2$ and y are real numbers, then*

$$\#\{n \in (t, 2t] : b_n b_{n+1} > z, b_n \leq y, \text{ and } b_{n+1} \leq y\} \ll \frac{t \log z}{z} + y^2.$$

Proof. The left–hand side is

$$\begin{aligned} &\leq \sum_{t<n\leq 2t} \sum_{\substack{b^2|n, c^2|(n+1) \\ bc>z, b\leq y, c\leq y}} 1 \quad \ll \sum_{\substack{bc>z \\ b\leq y, c\leq y}} \left(\frac{t}{b^2c^2} + 1\right) \\ &\ll y^2 + \sum_{bc\geq z} \frac{t}{b^2c^2}. \end{aligned} \tag{3}$$

Now the last sum in (3) is at most

$$t \sum_{r\geq z} d(r) r^{-2}, \tag{4}$$

where $d(r)$ denotes the number of divisors of r. Using the elementary estimate $\sum_{r\leq x} d(r) \ll x \log x$ and partial summation, we find that (4) is

$$\ll \frac{t \log z}{z}.$$

This completes the proof.

3. Proof of the theorem

We will bound

$$\#\{n \in (t, 2t] : b_{n-1} b_n^2 b_{n+1}^2 b_{n+2} > \tfrac{1}{3} t\}. \tag{5}$$

By Lemma 2, those n with any of $b_{n-1}, b_n, b_{n+1}, b_{n+2}$ greater than $t^{1/3}$ contribute $\ll t^{2/3}$. The remaining n all have $b_{n+j} \leq t^{1/3}$ for $-1 \leq j \leq 2$. By Lemma 3, those n with any of $b_{n-1}b_n, b_nb_{n+1}, b_{n+1}b_{n+2}$ greater than $t^{1/3}\log t$ contribute $\ll t^{2/3}$. The remaining n all have

$$b_{n-1}b_n, b_nb_{n+1}, b_{n+1}b_{n+2} \le t^{1/3}\log t.$$

Using the condition in (5), we see that

$$b_{n-1}b_n \cdot b_nb_{n+1} \cdot b_{n+1}b_{n+2} > \tfrac{1}{3}t,$$

so in fact the remaining n satisfy the stronger conditions

$$\tfrac{1}{3}t^{1/3}\log^{-2} t \le b_{n-1}b_n, b_nb_{n+1}, b_{n+1}b_{n+2} \le t^{1/3}\log t. \tag{6}$$

Now consider those n satisfying (6) with $b_n > t^{2/9}$. Then $b_{n-1}, b_{n+1} < t^{1/9}\log t$ and $b_{n+2} > \frac{1}{3}t^{2/9}\log^{-3} t$. In other words, these n have

$$b_n \le t^{1/3}, b_{n+2} \le t^{1/3} \text{ and } b_nb_{n+2} > \tfrac{1}{3}t^{4/9}\log^{-3} t.$$

By an easy variant of the argument giving Lemma 3, these n contribute

$$\ll t^{5/9}\log^4 t + t^{2/3} \ll t^{2/3}.$$

A similar argument can be used to get the same bound for those n with $b_{n+1} > t^{2/9}$.

The remaining n have $b_n, b_{n+1} \le t^{2/9}$. By (6), $b_{n-1} \ge \frac{1}{3}t^{1/9}\log^{-2} t$ and

$$\tfrac{1}{9}t^{4/9}\log^{-4} t \le b_{n-1}b_nb_{n+1} \le t^{5/9}\log t.$$

The number of such n is

$$\ll \sum_{\frac{1}{9}t^{4/9}\log^{-4} t \le m \le t^{5/9}\log t} \left(\frac{t}{m^2} + 1\right) d_3(m) \tag{7}$$

where $d_3(m)$ denotes the number of ways of writing m as a product of three factors. Using the trivial estimate $\sum_{m \le x} d_3(m) \ll x \log^2 x$ and partial summation, we see that (7) is

$$\ll t^{5/9} \log^6 t \ll t^{2/3}.$$

This concludes the proof.

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References

1. M. Filaseta, *The irreducibility of almost all Bessel polynomials*, J. No. Theory **27** (1987), 22–32.
2. M. Filaseta, *On an irreducibility theorem of I. Schur*, Acta Arith. **58** (1991), 251–272.
3. E. Grosswald, *Bessel Polynomials*, Lecture Notes in Math. Vol. 698, Springer-Verlag, Berlin, 1978.

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