Homework Review

Notes 1, Problem 1

Let a and b be positive integers, and write

$$\frac{a}{b} = m.d_1 d_2 \ldots d_k \overline{d_{k+1} d_{k+2} \ldots d_{k+r}}$$

where m is a positive integer, the d_j are digits, and r is chosen as small as possible. Prove that r divides $\phi(b)$ where ϕ is Euler's $\phi-$function.

$$m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

$$\phi(m) =$$

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$$\phi(m) = p_1^{e_1 - 1} \cdots p_r^{e_r - 1}$$

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$$\phi(m) = p_1^{e_1 - 1} \cdots p_r^{e_r - 1} (p_1 - 1) \cdots (p_r - 1)$$

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$$\phi(m) = p_1^{e_1 - 1} \cdots p_r^{e_r - 1}(p_1 - 1) \cdots (p_r - 1)$$

Remark: If $b' | b$, then $\phi(b') | \phi(b)$.

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WLOG, $k = 0$. Why?

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WLOG, $k = 0$. Why?

Otherwise consider $\frac{10^k a}{b}$.

$$\frac{a}{b} = m.\overline{d_1 d_2 \ldots d_r}$$

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$$\left(\, 10^r - 1 \,\right) \times \frac{a}{b} \in \mathbb{Z}$$

$$\left(10^{r}-1\right)\times\frac{a}{b}\in\mathbb{Z}$$

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$$b' |$$

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IDEA: r is the order of 10 modulo b'

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$$\implies r | \phi(b')$$

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$$\left(\,\mathbf{10}^{r} - \mathbf{1}\,\right) \times \frac{\boldsymbol{a'}}{\boldsymbol{b'}} \in \mathbb{Z}$$

Suppose $\frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\boldsymbol{a'}}{\boldsymbol{b'}}$ where $\mathbf{gcd}(\boldsymbol{a'}, \boldsymbol{b'}) = \mathbf{1}$. Then

$$\boldsymbol{b'} | \left(\,\mathbf{10}^{r} - \mathbf{1}\,\right).$$

IDEA: $\boldsymbol{r}$ is the order of 10 modulo $\boldsymbol{b'}$

$$\implies \boldsymbol{r} | \phi(\boldsymbol{b'})$$

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What's wrong with this IDEA?

Finish the proof by showing that if t is such that

$$b' | (10^t - 1),$$

then

$$\frac{a'}{b'} = m.\overline{d_1 d_2 \dots d_t}.$$

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So $t \geq r$.

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$$\gcd(a, b) = 1$$

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$$\implies k = 0$$

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$$360655737704{\color{red}9}18032786885245{\color{red}9}\ldots$$

Notes 1, Problem 2

Suppose $r = b - 1$.

(i) Prove that each of the digits $0, 1, \ldots, 9$ occurs among the digits $d_1, d_2, \ldots, d_r$ either

$$[(b-1)/10] \qquad \text{or} \qquad [(b-1)/10] + 1$$

times.

(ii) Prove that 0 occurs $[(b-1)/10]$ times among the digits $d_1, d_2, \ldots, d_r$.

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Note: $\boldsymbol{r = p - 1} \implies$ the order of 10 modulo $\boldsymbol{p}$ is $\boldsymbol{p - 1}$

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$$\frac{10^j a}{p} = d_1 d_2 \ldots d_j . d_{j+1} \ldots d_r d_1 d_2 \ldots$$

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r_j varies from 1 to $p - 1$

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Cases: $p = 10k + 1, p = 10k + 3,$
$p = 10k + 7, p = 10k + 9$

Case: $p = 10k + 1$

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$$k \qquad\qquad 3k - 2k \qquad\qquad k + 1$$

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$$(3k+2) - (2k+1) = k+1$$

Notes 1, Problem 3

Prove e^2 is irrational.

Main Tool: What power of $\mathbf{2}$ divides $\boldsymbol{n!}$?

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$$\left[\frac{\boldsymbol{n}}{\mathbf{2}}\right] + \left[\frac{\boldsymbol{n}}{4}\right] + \left[\frac{\boldsymbol{n}}{\mathbf{8}}\right] + \cdots$$

Main Tool: What power of $\mathbf{2}$ divides $\boldsymbol{n!}$?

$$\left[\frac{n}{2}\right] + \left[\frac{n}{4}\right] + \left[\frac{n}{8}\right] + \cdots \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots$$

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$$= n$$

When does $\mathbf{2}^{n-1}$ divide $\boldsymbol{n!}$?

Main Tool: What power of $\mathbf{2}$ divides $\boldsymbol{n!}$?

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