Homework Review

Notes 1, Problem 1

Let a and b be positive integers, and write

$$\frac{a}{b} = m.d_1d_2\ldots d_k\overline{d_{k+1}d_{k+2}\ldots d_{k+r}}$$

where m is a positive integer, the d_j are digits, and r is chosen as small as possible. Prove that r divides $\phi(b)$ where ϕ is Euler's $\phi-$function.

$$m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

$$\phi(m) =$$

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$$\phi(m) = p_1^{e_1 - 1} \cdots p_r^{e_r - 1} (p_1 - 1) \cdots (p_r - 1)$$

Remark: If $b' | b$, then $\phi(b') | \phi(b)$.

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WLOG, $k = 0$. Why?

$$\frac{a}{b} = m.d_1 d_2 \dots d_k \overline{d_{k+1} d_{k+2} \dots d_{k+r}}$$

WLOG, $k = 0$. Why?

Otherwise consider $\dfrac{10^k a}{b}$.

$$\frac{a}{b} = m.\overline{d_1 d_2 \ldots d_r}$$

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$$\left(10^r - 1\right) \times \frac{a}{b} = m d_1 d_2 \ldots d_r - m$$

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$$\left(10^r - 1\right) \times \frac{a}{b} = m d_1 d_2 \ldots d_r - m$$

$$\left(\, 10^r - 1 \,\right) \times \frac{a}{b} \in \mathbb{Z}$$

$$\left(\mathbf{10}^{\boldsymbol{r}} - \mathbf{1}\right) \times \frac{\boldsymbol{a}}{\boldsymbol{b}} \in \mathbb{Z}$$

$$\left(10^r - 1 \right) \times \frac{a}{b} \in \mathbb{Z}$$

Suppose $\frac{a}{b} = \frac{a'}{b'}$ where $\gcd(a', b') = 1$.

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$$b' |$$

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Suppose $\frac{a}{b} = \frac{a'}{b'}$ where $\gcd(a', b') = 1$. Then

$$b' | \left(10^r - 1\right).$$

IDEA: r is the order of 10 modulo b'

$$\left(10^r - 1\right) \times \frac{a'}{b'} \in \mathbb{Z}$$

Suppose $\frac{a}{b} = \frac{a'}{b'}$ where $\gcd(a', b') = 1$. Then

$$b' \mid \left(10^r - 1\right).$$

IDEA: r is the order of 10 modulo b'

$$\implies r \mid \phi(b')$$

$$\left(10^r - 1\right) \times \frac{a'}{b'} \in \mathbb{Z}$$

Suppose $\frac{a}{b} = \frac{a'}{b'}$ where $\gcd(a', b') = 1$. Then

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What's wrong with this IDEA?

Finish the proof by showing that if t is such that

$$b' | (10^t - 1),$$

then

$$\frac{a'}{b'} = m.\overline{d_1 d_2 \ldots d_t}.$$

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then

$$\frac{a'}{b'} = m.\overline{d_1 d_2 \ldots d_t}.$$

So $t \geq r$.

Notes 1, Problem 2

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$$r = b - 1 \implies$$

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$$b \neq 2$$

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$$\gcd(a, b) = 1$$

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$$\implies k = 0$$

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$$\frac{1}{11} = 0.090909\ldots$$

$$\frac{1}{61} = 0.0163934426229508196721311475409836065573770491803278688524590\ldots$$

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$$\frac{1}{11} = 0.090909\ldots$$

$$\frac{1}{61} = 0.0163934426229508196721311475409836065573770491803278688524 59\ldots$$

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Examples:

$$\frac{1}{11} = 0.090909\ldots$$

$$\frac{1}{61} = 0.016393\textcolor{red}{44}262295081967213114\textcolor{red}{4}75\textcolor{red}{4}098$$
$$360655737704\textcolor{red}{4}918032786885 2\textcolor{red}{4}59\ldots$$

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Examples:

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Examples:

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$$\frac{1}{61} = 0.01{\color{red}6}393442{\color{red}6}22950819{\color{red}6}7213114754098$$
$$3{\color{red}6}0{\color{red}6}5573770491803278{\color{red}6}8852459\ldots$$

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Examples:

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$$360655737704918{\color{red}8}0327{\color{red}8}6{\color{red}8}{\color{red}8}52459\ldots$$

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Notes 1, Problem 2

Suppose $\boldsymbol{r = b - 1}$.

(i) Prove that each of the digits $\mathbf{0, 1, \ldots, 9}$ occurs among the digits $\boldsymbol{d_1, d_2, \ldots, d_r}$ either

$$\boldsymbol{[(b-1)/10]} \qquad \text{or} \qquad \boldsymbol{[(b-1)/10]+1}$$

times.

(ii) Prove that 0 occurs $\boldsymbol{[(b-1)/10]}$ times among the digits $\boldsymbol{d_1, d_2, \ldots, d_r}$.

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The values of $10^j a$ modulo p are distinct for $0 \le j \le p-2$ and are congruent to $1, 2, \ldots, p-1$ in some order.

$$b = p, \text{ a prime}$$

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$$10^j a = pq_j + r_j$$

$$\boldsymbol{b = p},\ \text{a prime}$$

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$$\boldsymbol{10^j a = p q_j + r_j \implies \frac{10^j a}{p} = q_j + \frac{r_j}{p}}$$

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$$\frac{10^j a}{p} = d_1 d_2 \ldots d_j . d_{j+1} \ldots d_r d_1 d_2 \ldots$$

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r_j varies from 1 to $p - 1$

$$\frac{10^j a}{p} = \underbrace{d_1 d_2 \ldots d_j}_{q_j} \underbrace{.d_{j+1} \ldots d_r d_1 d_2 \ldots}_{r_j/p}$$

$$0 \leq \frac{r_j}{p} < \frac{1}{10} \implies$$

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$$\frac{2}{10} \leq \frac{r_j}{p} < \frac{3}{10} \implies d_{j+1} = 2$$

$$\frac{10^j a}{p} = \underbrace{d_1 d_2 \dots d_j}_{q_j} . \underbrace{d_{j+1} \dots d_r d_1 d_2 \dots}_{r_j/p}$$

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$$\vdots \qquad\qquad \vdots$$

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$$1 \leq r_j \leq p - 1$$

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$d_{j+1} = 2$ for $\left[\frac{3p}{10}\right] - \left[\frac{2p}{10}\right]$ values of $j \in [1, p-1]$

$$\boldsymbol{d_{j+1} = 2} \text{ for } \left[\frac{\boldsymbol{3p}}{\boldsymbol{10}}\right] - \left[\frac{\boldsymbol{2p}}{\boldsymbol{10}}\right] \text{ values of } \boldsymbol{j} \in [\boldsymbol{1, p-1}]$$

$$\boldsymbol{d_{j+1} = 2} \text{ for } \left[\frac{\mathbf{3}\boldsymbol{p}}{\mathbf{10}}\right] - \left[\frac{\mathbf{2}\boldsymbol{p}}{\mathbf{10}}\right] \text{ values of } \boldsymbol{j} \in [\mathbf{1}, \boldsymbol{p} - \mathbf{1}]$$

To Show:

$$d_{j+1} = 2 \text{ for } \left\lceil \frac{3p}{10} \right\rceil - \left\lceil \frac{2p}{10} \right\rceil \text{ values of } j \in [1, p-1]$$

To Show:

$$\left\lceil \frac{p-1}{10} \right\rceil \leq \left\lceil \frac{3p}{10} \right\rceil - \left\lceil \frac{2p}{10} \right\rceil \leq \left\lceil \frac{p-1}{10} \right\rceil + 1$$

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To Show:

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Cases: $p = 10k+1, p = 10k+3,$
$p = 10k+7, p = 10k+9$

Case: $p = 10k + 1$

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Case: $p = 10k + 1$

$$\left[\frac{p-1}{10}\right] \leq \left[\frac{3p}{10}\right] - \left[\frac{2p}{10}\right] \leq \left[\frac{p-1}{10}\right] + 1$$

$\downarrow$

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$$k$$

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$$\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$k \qquad\qquad 3k - 2k \qquad\qquad k + 1$$

Case: $p = 10k + 7$

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$$\downarrow$$

$$(3k+2) - (2k+1)$$

Case: $p = 10k + 7$

$$\left[\frac{p-1}{10}\right] \leq \left[\frac{3p}{10}\right] - \left[\frac{2p}{10}\right] \leq \left[\frac{p-1}{10}\right] + 1$$

$$\downarrow$$

$$(3k+2) - (2k+1) = k+1$$

Notes 1, Problem 3

Prove e^2 is irrational.

Main Tool: What power of $\mathbf{2}$ divides $\boldsymbol{n!}$?

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$$\left[\frac{\boldsymbol{n}}{\mathbf{2}}\right] + \left[\frac{\boldsymbol{n}}{\mathbf{4}}\right] + \left[\frac{\boldsymbol{n}}{\mathbf{8}}\right] + \cdots$$

Main Tool: What power of $\mathbf{2}$ divides $\boldsymbol{n!}$?

$$\left[\frac{n}{2}\right] + \left[\frac{n}{4}\right] + \left[\frac{n}{8}\right] + \cdots \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots$$

Main Tool: What power of $\mathbf{2}$ divides $\boldsymbol{n!}$?

$$\left[\frac{n}{2}\right] + \left[\frac{n}{4}\right] + \left[\frac{n}{8}\right] + \cdots \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots$$

$$= n$$

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