TEST 2, MATH 785

1. (a) Define what it means for a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ to be *uniformly distributed modulo* 1.

   (b) State Weyl's Theorem associated with the sequence $\{n\alpha\}_{n=1}^{\infty}$.

   (c) Let $A(x)$ denote the number of positive integers $n \le x$ such that the leading digit of 2^n is 1. Explain why Weyl's Theorem implies that
   $$\lim_{x\to\infty} \frac{A(x)}{x} = \log_{10} 2.$$
   Your explanation should be self-contained.

2. Using the lemma below, prove Weyl's theorem.

   Lemma: *Let $\alpha \in \mathbb{R}$ with α irrational. Suppose $f : \mathbb{R} \to \mathbb{C}$ satisfies:*

   (i) $f(x)$ is continuous over the reals.
   (ii) $f(x + 2\pi) = f(x)$ for all real x.

   Then $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(2\pi r\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx.$

3. Using the theorem below, prove that $2^a = 3^b + 1$ has finitely many solutions in integers a and b. Clearly indicate in your argument the values of r, d, A, B, the α_j's, and the β_j's.

   Theorem: *Let $\alpha_1, \dots, \alpha_r$ be non-zero algebraic numbers with degrees at most d and heights at most A. Let $\beta_0, \beta_1, \dots, \beta_r$ be algebraic numbers with degrees at most d and heights at most $B > 1$. Suppose that*
   $$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_r \log \alpha_r \ne 0.$$
   Then there are numbers $C = C(r, d) > 0$ and $w = w(r) \ge 1$ such that
   $$|\Lambda| > B^{-C(\log A)^w}.$$

4. Give a short answer for each of the following. You may make use of results from class.

   (a) Let $\gamma = \dfrac{1}{\sqrt{3}}$. Calculate with justification **den**(γ) (the denominator, as defined in class, of the algebraic number γ).

   (b) Explain why if α is an algebraic number and d is a rational integer for which $d\alpha$ is an algebraic integer, then the denominator of α divides d.

   (c) Explain why the number $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{3\cdot 2^{k+1} + 7\cdot 2^k}}$ is transcendental.