MATH 785, TEST 1

1. Determine whether each of the following is true or false. Give brief justifications for your conclusions. (Note that "true" is intended to mean "true always" and "false" is intended to mean "false at least in some case".)

   (a) The sum of two algebraic numbers is algebraic.

   (b) The sum of two transcendental numbers is transcendental.

   (c) The product of two transcendental numbers is transcendental.

   (d) The sum of an irrational number and a rational number is irrational.

   (e) An irrational number to an irrational power is irrational.

2. From Lindemann's theorem, if α_1 and α_2 are distinct algebraic numbers and β_1 and β_2 are non-zero algebraic numbers, then $\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} \neq 0$. Explain why this implies that π is transcendental. You may use that the product of two algebraic numbers is algebraic *provided* you clearly indicate when you are using this fact.

3. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers having the property that for every N there is an n for which $a_{n+1}/a_n > N$. Prove that $\sum_{n=1}^{\infty} 2^{-a_n}$ is a Liouville number. (Do not use any stated but unproven information from the notes.)

4. To prove e is transcendental, we assumed that e was a root of a polynomial

$$g(x) = b_0 + b_1 x + \cdots + b_r x^r \in \mathbb{Z}[x]$$

   with $b_0 \neq 0$. For p a sufficiently large prime, we considered

$$f(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-r)^p,$$

$$I(t) = e^t \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(t),$$

   and

$$J = b_0 I(0) + b_1 I(1) + \cdots + b_r I(r).$$

   where n is the degree of $f(x)$.

   (a) Prove that $|J| \geq (p-1)!$.

   (b) Recall that $\sum_{k=0}^{\infty} x^k/k!$ converges to e^x for all real x. Explain why this implies $n! > n^n/e^n$ for all positive integers n.

   (c) We also showed that $|J| \leq c_1 c_2^p$ for some constants c_1 and c_2 independent of p. Using both (a) and (b), explain how this leads to a contradiction.

5. (a) Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence of positive integers such that $a_n < a_{n+1} \leq 10a_n$ for all $n \geq 0$. Consider the number $\alpha = 0.a_1a_2a_3\dots$ formed by concatenating the numbers a_n (for example, if $a_n = 2^n$, then $\alpha = 0.248163264128\dots$). Prove that α is irrational. (Hint: Assume α is rational. If r is the length of the periodic part of α, first explain why for every sufficiently large integer k there is an a_n with exactly kr digits. What can one say about a_{n+1}?)

(b) Suppose the condition $a_n < a_{n+1} \leq 10a_n$ for all $n \geq 0$ is replaced by $a_n < a_{n+1} \leq 10a_n + 1$ for all $n \geq 0$ in part (a). Explain why the conclusion that α is irrational no longer holds.