## 5 The Beginning of Transcendental Numbers

We have defined a transcendental number (see Definition 3 of the Introduction), but so far we have only established that certain numbers are irrational. We now turn to the beginnings of transcendental numbers. Our first theorem is

Theorem 11. Transcendental numbers exist.
Like many of our results so far, this will of course be a consequence of later results. The first proof that there exist transcendental numbers was given by Liouville. Before we give his proof, we give a proof due to Cantor.

Proof 1. The essence of this proof is that the real algebraic numbers are countable whereas the set of all real numbers is uncountable, so there must exist real transcendental numbers. Define

$$
P(n)=\left\{f(x)=\sum_{j=0}^{n} a_{j} x^{j} \in \mathbb{Z}[x]: 1 \leq \sum_{j=0}^{n}\left|a_{j}\right| \leq n\right\} .
$$

Observe that $P(n)$ is finite. Also, every non-zero polynomial in $\mathbb{Z}[x]$ belongs to some $P(n)$. By considering the real roots of polynomials in $P(1), P(2), \ldots$ (at the $k$ th stage, consider the real roots of polynomials in $P(k)$ which have not occurred as a root of a polynomial in $P(j)$ for $j<k$, we can order the algebraic numbers; hence, they are countable. Next, give the usual proof that the real numbers are uncountable.

For Liouville's proof, we define
Definition 4. A real number $\alpha$ is a Liouville number if for every positive integer n, there are integers $a$ and $b$ with $b>1$ such that

$$
0<\left|\alpha-\frac{a}{b}\right|<\frac{1}{b^{n}} .
$$

In a moment, we will show that Liouville numbers exist. The second proof of Theorem 11 will then follow from our next result.

Theorem 12. All Liouville numbers are transcendental.
Lemma 1. Let $\alpha$ be an irrational number which is a root of $f(x)=\sum_{j=0}^{n} a_{j} x^{j} \in \mathbb{Z}[x]$ with $f(x) \not \equiv 0$. Then there is a constant $A=A(\alpha)>0$ such that if $a$ and $b$ are integers with $b>0$, then

$$
\begin{equation*}
\left|\alpha-\frac{a}{b}\right|>\frac{A}{b^{n}} . \tag{6}
\end{equation*}
$$

Proof. Let $M$ be the maximum value of $\left|f^{\prime}(x)\right|$ on $[\alpha-1, \alpha+1]$. Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}$ be the distinct roots of $f(x)$ which are different from $\alpha$. Fix

$$
A<\min \left\{1,1 / M,\left|\alpha-\alpha_{1}\right|,\left|\alpha-\alpha_{2}\right|, \ldots,\left|\alpha-\alpha_{m}\right|\right\} .
$$

Assume (6) does not hold for some $a$ and $b$ integers with $b>0$. Then

$$
\left|\alpha-\frac{a}{b}\right| \leq \frac{A}{b^{n}} \leq A<\min \left\{1,\left|\alpha-\alpha_{1}\right|,\left|\alpha-\alpha_{2}\right|, \ldots,\left|\alpha-\alpha_{m}\right|\right\} .
$$

Hence,

$$
\frac{a}{b} \in[\alpha-1, \alpha+1] \quad \text { and } \quad \frac{a}{b} \notin\left\{\alpha_{1}, \ldots, \alpha_{m}\right\} .
$$

By the Mean Value Theorem, there is an $x_{0}$ between $a / b$ and $\alpha$ such that

$$
f(\alpha)-f(a / b)=(\alpha-a / b) f^{\prime}\left(x_{0}\right)
$$

so that

$$
\left|\alpha-\frac{a}{b}\right|=\left|\frac{f(a / b)-f(\alpha)}{f^{\prime}\left(x_{0}\right)}\right|=\frac{|f(a / b)|}{\left|f^{\prime}\left(x_{0}\right)\right|} .
$$

Since $f(a / b) \neq 0$, we deduce that

$$
|f(a / b)|=\left|\sum_{j=0}^{n} a_{j} a^{j} b^{n-j}\right| / b^{n} \geq 1 / b^{n}
$$

Thus, since $\left|f^{\prime}\left(x_{0}\right)\right| \leq M$, we obtain

$$
\left|\alpha-\frac{a}{b}\right| \geq \frac{1}{M b^{n}}>\frac{A}{b^{n}} \geq\left|\alpha-\frac{a}{b}\right|,
$$

giving a contradiction. Thus, the lemma follows.
Proof of Theorem 12. Let $\alpha$ be a Liouville number. First, we show that $\alpha$ must be irrational. Assume $\alpha=c / d$ for some integers $c$ and $d$ with $d>0$. Let $n$ be a positive integer with $2^{n-1}>d$. Then for any integers $a$ and $b$ with $b>1$ and $a / b \neq c / d$, we have that

$$
\left|\alpha-\frac{a}{b}\right|=\left|\frac{c}{d}-\frac{a}{b}\right| \geq \frac{1}{b d}>\frac{1}{2^{n-1} b} \geq \frac{1}{b^{n}} .
$$

It follows from the definition of a Liouville number that $\alpha$ is not a Liouville number, giving a contradiction. Thus, $\alpha$ is irrational.

Now, assume $\alpha$ is an irrational algebraic number. By the lemma, there exist a real number $A>0$ and a positive integer $n$ such that (6) holds for all integers $a$ and $b$ with $b>0$. Let $r$ be a positive integer for which $2^{r} \geq 1 / A$. Since $\alpha$ is a Liouville number, there are integers $a$ and $b$ with $b>1$ such that

$$
\left|\alpha-\frac{a}{b}\right|<\frac{1}{b^{n+r}} \leq \frac{1}{2^{r} b^{n}} \leq \frac{A}{b^{n}} .
$$

This contradicts (6) and, hence, establishes that $\alpha$ is transcendental.
Example: We show that $\alpha=\sum_{j=0}^{\infty} 1 / 2^{j!}$ is a Liouville number. First, observe that the binary expansion of $\alpha$ has arbitrarily long strings of 0 's and so it cannot be rational. Fix a positive integer $n$ and consider $\frac{a}{b}=\sum_{j=0}^{n} \frac{1}{2^{j!}}$ with $a$ and $b=2^{n!}>1$ integers. Then

$$
0<\left|\alpha-\frac{a}{b}\right|=\sum_{j=n+1}^{\infty} \frac{1}{2^{j!}}<\sum_{j=(n+1)!}^{\infty} \frac{1}{2^{j}}=\frac{1}{2^{(n+1)!-1}} \leq \frac{1}{2^{n(n!)}}=\frac{1}{b^{n}} .
$$

This proves that $\alpha$ is Liouville and also gives a second proof of Theorem 11.

There are stronger versions of Theorem 12. In particular, we note (without proof) that
Theorem 13 (Thué-Siegel-Roth). Let $\alpha$ be an algebraic number with $\alpha \notin \mathbb{Q}$. Let $\epsilon>0$. Then there are at most finitely many pairs of integers $(a, b)$ with $b>0$ such that

$$
\left|\alpha-\frac{a}{b}\right|<\frac{1}{b^{2+\epsilon}} .
$$

It is not known whether or not the right-hand side above can be replaced by $A / b^{2}$ where $A=$ $A(\alpha)$ is a positive constant depending only on $\alpha$.

Theorem 14. The set of Liouville numbers in $[0,1]$ has measure 0 .
Proof. Let $\epsilon>0$. It suffices to show that the (Lebesgue) measure of the Liouville numbers in $[0,1]$ is $<\epsilon$. Let $n$ be a positive integer for which $\sum_{b=2}^{\infty} 4 / b^{n-1}<\epsilon$ (observe that this is possible since $\left.\sum_{b=2}^{\infty} 4 / b^{n-1}<\left(4 / 2^{n-3}\right) \sum_{b=2}^{\infty} 1 / b^{2}\right)$. If $\alpha$ is a Liouville number in [ 0,1 ], then there are integers $a$ and $b$ with $b>1$ such that

$$
\left|\alpha-\frac{a}{b}\right|<\frac{1}{b^{n}} .
$$

Since the right-hand side is $\leq 1 / 2$ and $\alpha \in[0,1]$, we deduce that $a / b \in(-1 / 2,3 / 2)$ so that $-b / 2<a<3 b / 2$ ( $a$ is in an open interval of length $2 b$ ). In particular, for a given integer $b>1$, there are $\leq 2 b$ possible values of $a$ for which the above inequality can hold. For each $b>1$, we get that $\alpha$ must be in one of $\leq 2 b$ intervals of length $2 / b^{n}$. Thus, the measure of the Liouville numbers must be

$$
\leq \sum_{b=2}^{\infty} \frac{4 b}{b^{n}}<\epsilon,
$$

giving the desired result.
Corollary 1. There are transcendental numbers which are not Liouville numbers.
We now turn to a related discussion. Suppose that we have an sequence of positive integers $\left\{a_{k}\right\}_{k=1}^{\infty}$ with $a_{1}<a_{2}<\cdots$. When will the argument given in the example above lead to a proof that $\sum_{k=0}^{\infty} 1 / 2^{a_{k}}$ is transcendental? It is not too difficult to see that what one essentially wants is that $a_{k}$ increases "fast enough" where fast enough means that

$$
\liminf _{k \rightarrow \infty} \frac{a_{k+1}}{a_{k}}=\infty .
$$

On the other hand, it can be shown that sequences which increase much slower also lead to transcendental numbers, an observation apparently first noticed by Erdős. The next theorem illustrates the basic idea by showing a certain number of this form is transcendental; it can be shown also that this number is not a Liouville number.

Theorem 15. The number $\sum_{k=0}^{\infty} 1 / 2^{2^{k}}$ is transcendental.
To prove Theorem 15, for positive integers $k$ and $m$, we define $c(k, m)$ to be the number of $m$-tuples $\left(j_{1}, \ldots, j_{m}\right)$ of non-negative integers for which

$$
k=2^{j_{1}}+2^{j_{2}}+\cdots+2^{j_{m}} .
$$

Lemma 1. With the notation above, $c(k, m) \leq m^{2 m}$.
Proof. We do induction on $m$. For $m=1, c(k, m) \in\{0,1\}$ for all $k$, so the result is clear. Suppose $m>1$. If $k$ has more than $m$ non-zero binary digits, then it is not too hard to see that $c(k, m)=0$ (note that the sum of two like powers of 2 is a power of 2 ). If $k$ has exactly $m$ non-zero binary digits, then $c(k, m)=m!\leq m^{m} \leq m^{2 m}$ ( $j_{1}$ can correspond to any one of the $m$ non-zero digits, $j_{2}$ to any one of the remaining $m-1$ non-zero digits, and so on). If $k$ has fewer than $m$ non-zero binary digits, then for some integers $r$ and $s$ with $1 \leq r<s \leq m, j_{r}=j_{s}$. Since in this case, $2^{j_{r}}+2^{j_{s}}=2^{j_{r}+1}$, we deduce that

$$
c(k, m) \leq\binom{ m}{2} c(k, m-1) \leq m^{2}(m-1)^{2 m-2} \leq m^{2 m}
$$

establishing the lemma.
Lemma 2. Let $t$ and $m$ be positive integers. Then $c(k, m)=0$ for every integer $k \in\left(2^{t+1}+\cdots+\right.$ $\left.2^{t+m}, 2^{t+m+1}\right)$.

Proof. This follows since each such $k$ has $>m$ non-zero binary digits (see the proof of Lemma $1)$.

Proof of Theorem 15. Let $\alpha=\sum_{k=0}^{\infty} 1 / 2^{2^{k}}$. Let $m$ be a positive integer. Then the definition of $c(k, m)$ implies that

$$
\alpha^{m}=\left(\sum_{k=0}^{\infty} \frac{1}{2^{2^{k}}}\right)^{m}=\sum_{k=1}^{\infty} \frac{c(k, m)}{2^{k}} .
$$

Let $t$ be a positive integer. Using Lemma 2 and then Lemma 1, we obtain

$$
\begin{aligned}
\left\{2^{2^{t+1}+\cdots+2^{t+m}} \alpha^{m}\right\} & \leq 2^{2^{t+1}+\cdots+2^{t+m}} \sum_{k=2^{t+1}+\cdots+2^{t+m}+1}^{\infty} \frac{c(k, m)}{2^{k}} \\
& \leq 2^{2^{t+1}+\cdots+2^{t+m}} \sum_{k=2^{t+m+1}}^{\infty} \frac{c(k, m)}{2^{k}} \\
& \leq 2^{2^{t+1}+\cdots+2^{t+m}} m^{2 m} \sum_{k=2^{t+m+1}}^{\infty} \frac{1}{2^{k}} \\
& \leq 2^{2^{t+1}+\cdots+2^{t+m}} m^{2 m} 2^{-2^{t+m+1}+1}=2^{1-2^{t+1}} m^{2 m}
\end{aligned}
$$

If we view $m$ as being fixed and let $t$ approach infinity, we see that the binary expansion of $\alpha^{m}$ has arbitrarily long strings of 0 's. By the same reasoning, we see that more generally if $b_{0}, b_{1}, \ldots, b_{m}$ are non-negative integers, then the binary expansion of $b_{0}+b_{1} \alpha+\cdots+b_{m} \alpha^{m}$ has arbitrarily long strings of 0 's. In fact, if $b_{0}, b_{1}, \ldots, b_{m}$ and $c_{0}, c_{1}, \ldots, c_{m}$ are non-negative integers and if we have the following binary expansions

$$
\left\{b_{0}+b_{1} \alpha+\cdots+b_{m} \alpha^{m}\right\}=\left(0 . d_{1} d_{2} d_{3} \ldots\right)_{2}
$$

and

$$
\left\{c_{0}+c_{1} \alpha+\cdots+c_{m} \alpha^{m}\right\}=\left(0 \cdot d_{1}^{\prime} d_{2}^{\prime} d_{3}^{\prime} \ldots\right)_{2}
$$

then for any positive integer $N$, there is a positive integer $j$ for which

$$
d_{j+1}=d_{j+2}=\cdots=d_{j+N}=d_{j+1}^{\prime}=d_{j+2}^{\prime}=\cdots=d_{j+N}^{\prime}=0
$$

Furthermore, we can take $N$ and $j$ so that

$$
N=2^{t} \quad \text { and } \quad j=2^{t+1}+\cdots+2^{t+m}
$$

with $t$ an integer as large (but perhaps not as small) as we wish.
Now, assume $\alpha$ is a root of $f(x)=\sum_{j=0}^{n} a_{j} x^{j} \in \mathbb{Z}[x]$ with $a_{n}>0$ (which would be possible if $\alpha$ were algebraic). Then we can write $f(x)$ in the form

$$
f(x)=\sum_{j=0}^{n} b_{j} x^{j}-\sum_{j=0}^{n} c_{j} x^{j},
$$

where the $b_{j}$ and $c_{j}$ are non-negative integers with $b_{j} c_{j}=0$ for each $j$. In particular, $b_{n}=a_{n}>0$ and $c_{n}=0$. Take $m=n-1$, and let $N=2^{t}$ where $t$ is a positive integer to be chosen momentarily and $j=j(t)$ is as above. Then

$$
2^{j+2^{t-2}} \sum_{i=0}^{n-1} b_{i} \alpha^{i} \quad \text { and } \quad 2^{j+2^{t-2}} \sum_{i=0}^{n-1} c_{i} \alpha^{i}
$$

both differ from an integer by $\leq 1 / 2^{N-2^{t-2}}=1 / 2^{2^{t-1}+2^{t-2}}$. If we write

$$
2^{j+2^{t-2}} \sum_{i=0}^{n-1} b_{i} \alpha^{i}=m_{1}+\theta_{1} \quad \text { and } \quad 2^{j+2^{t-2}} \sum_{i=0}^{n-1} c_{i} \alpha^{i}=m_{2}+\theta_{2},
$$

where $m_{1}$ and $m_{2}$ are the greatest integers in the above expressions, then we see that since $f(\alpha)=$ 0 ,

$$
\begin{equation*}
2^{j+2^{t-2}} b_{n} \alpha^{n}=m_{3}+\theta_{3} \quad \text { with } m_{3} \in \mathbb{Z} \text { and }\left|\theta_{3}\right|=\left|\theta_{1}-\theta_{2}\right| \leq \frac{1}{2^{2^{t-1}+2^{t-2}}} \tag{7}
\end{equation*}
$$

On the other hand, since $j$ has the form given above, we get from the definition of $c(k, m)$ that $c\left(j+2^{t-1}, n\right) \geq 1$ and $c(k, n)=0$ for all $k \in\left(j+2^{t-2}, j+2^{t-1}\right)$. For $t$ sufficiently large, we have that

$$
\begin{aligned}
2^{j+2^{t-2}} b_{n} \sum_{k=j+2^{t-2}+1}^{\infty} \frac{c(k, n)}{2^{k}} & =2^{j+2^{t-2}} b_{n} \sum_{k=j+2^{t-1}}^{\infty} \frac{c(k, n)}{2^{k}} \\
& \leq 2^{j+2^{t-2}} b_{n} n^{2 n} 2^{-j-2^{t-1}+1}=2^{1-2^{t-2}} b_{n} n^{2 n}<\frac{1}{2^{t^{t-3}}}
\end{aligned}
$$

and

$$
2^{j+2^{t-2}} b_{n} \sum_{k=j+2^{t-2}+1}^{\infty} \frac{c(k, n)}{2^{k}} \geq 2^{j+2^{t-2}} b_{n} 2^{-j-2^{t-1}}=2^{-2^{t-2}} b_{n} \geq \frac{1}{2^{2^{t-2}}} .
$$

It follows that (7) cannot hold, giving a contradiction and completing the proof.

## Homework:

1. Prove that the set of Liouville numbers in $[0,1]$ is uncountable.
