(1) The number 2 can appear in any one of 4 positions for a 4-permutation. The remaining three positions correspond to a 3-permutation of the five numbers from the set $\{1, 3, 4, 5, 6\}$. There are $5 \times 4 \times 3 = 60$ such 3-permutations. Hence, the answer is $4 \times 60 = 240$.

(2) The answer is r where r is the largest positive integer such that 2^r divides $50!$. Thus, the answer is

$$
\begin{aligned}
r &= \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] \\
&= 25 + 12 + 6 + 3 + 1 \\
&= 47.
\end{aligned}
$$

(4) (b) The number of subsets of $\{1, 2, \ldots, n\}$ of size k is $\binom{n}{k}$. It follows that

$$|\mathcal{A}_1| + |\mathcal{A}_2| + \cdots + |\mathcal{A}_r| = \sum_{k=0}^{n} k \binom{n}{k}.$$

We evaluated this last expression in class (see Notes 6). The answer is $n2^{n-1}$.

(5) We take $x = 1/2$ in the binomial theorem

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

to obtain

$$(3/2)^n = \sum_{k=0}^{n} \frac{\binom{n}{k}}{2^k} = 1 + \sum_{k=1}^{n} \frac{\binom{n}{k}}{2^k}$$

(since the summand $\binom{n}{k}/2^k$ equals 1 when $k = 0$). Subtracting 1 from both sides of the equation, we obtain

$$\sum_{k=1}^{n} \frac{\binom{n}{k}}{2^k} = (3/2)^n - 1.$$

(6) Observe that

$$\int_0^1 x^k \, dx = \frac{x^{k+1}}{k+1}\bigg|_{x=0}^{x=1} = \frac{1}{k+1}$$

and

$$\int_0^1 (x+1)^n \, dx = \frac{(x+1)^{n+1}}{n+1}\bigg|_{x=0}^{x=1} = \frac{2^{n+1}-1}{n+1}.$$

By integrating from 0 to 1 both sides of the binomial theorem $\sum_{k=0}^{n} \binom{n}{k} x^k = (x+1)^n$, we deduce that

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = \frac{2^{n+1}-1}{n+1},$$

which is equivalent to what was to be shown.