MATH 574: TEST 1

Name _____

Points: Problems (1), (2), and (3) are worth 15 points each, and Problems (4), (5), and (6) are worth 18 points each. Your correctly spelled name or nickname above is worth 1 point.

(1) Suppose we want to use induction to prove

 $8^n - 7n - 1$ is divisible by 49

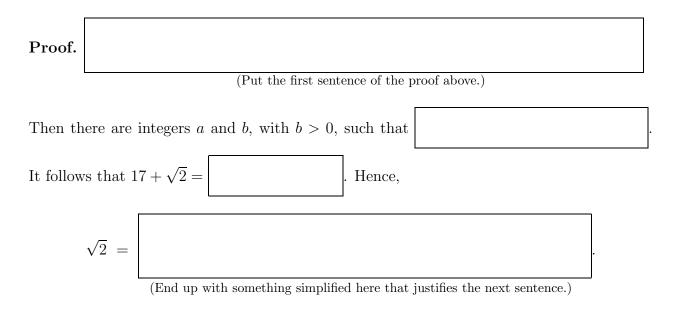
for all positive integers n.

(a) What would be the "induction hypothesis" in the proof? (Don't forget to use the word "some" or the word "all" below.)

(b) After stating the induction hypothesis in the proof, what should the goal be? In other words, what should we be trying to establish? Be precise.

(2) Prove that $\log_2 3$ is irrational.

(3) We showed in class that $\sqrt{2}$ is irrational. Using this information, complete the boxes below so as to complete a proof that $\sqrt{17 + \sqrt{2}}$ is irrational.



This is a contradiction since we showed in class that $\sqrt{2}$ is irrational. Therefore, $\sqrt{17 + \sqrt{2}}$

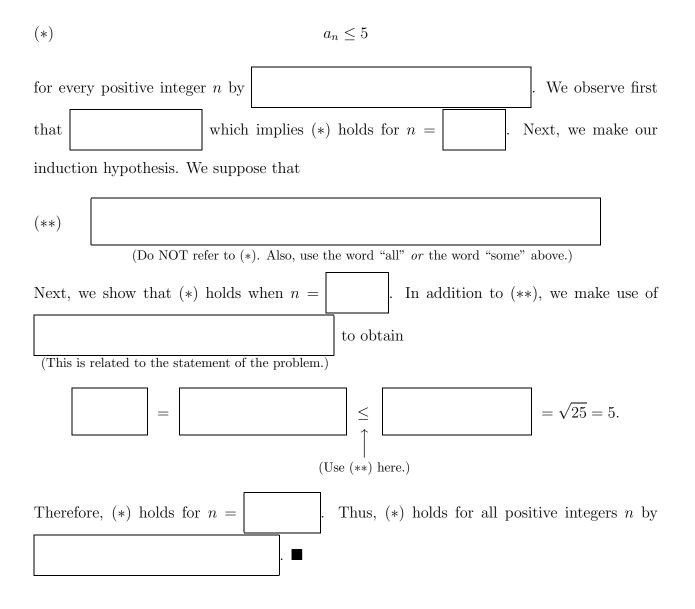
is irrational. \blacksquare

(4) Let $a_1 = \sqrt{20}$, and let

 $a_{n+1} = \sqrt{20 + a_n}$ for each positive integer *n*.

For example, $a_2 = \sqrt{20 + \sqrt{20}}$ and $a_3 = \sqrt{20 + \sqrt{20 + \sqrt{20}}}$. Complete the boxes below to give a proof that $a_n \leq 5$ for every positive integer n.

Proof. We prove



(5) Let $k_1, k_2, k_3, \ldots, k_{98}, k_{99}$ be some ordering of the numbers $1, 2, 3, \ldots, 98, 99$. (For example, maybe $k_1 = 2, k_2 = 1$, and $k_j = 102 - j$ for $3 \le j \le 99$, but maybe not. That's just an example.) Complete the boxes below and finish the proof to show that there must be some positive integer $i \le 99$ for which $k_i - i$ is even.

Proof. Assume that there is no positive integer $i \leq 99$ for which $k_i - i$ is even. Then the numbers $k_1 - 1, k_2 - 2, k_3 - 3, \ldots, k_{98} - 98$, and $k_{99} - 99$ are all odd. Since the sum of an odd number of odd numbers is ______, we deduce that $(k_1 - 1) + (k_2 - 2) + (k_3 - 3) + \cdots + (k_{98} - 98) + (k_{99} - 99)$ is ______.

(A word goes here, not a number.)

Since $k_1, k_2, k_3, \ldots, k_{98}, k_{99}$ is some ordering of the numbers $1, 2, 3, \ldots, 98, 99$, we obtain

$$k_1 + k_2 + k_3 + \dots + k_{98} + k_{99} =$$
(You do not need an exact value here.)

Therefore,

$$(k_{1} - 1) + (k_{2} - 2) + (k_{3} - 3) + \dots + (k_{98} - 98) + (k_{99} - 99)$$

$$= (k_{1} + k_{2} + k_{3} + \dots + k_{98} + k_{99}) - ($$

$$=$$
(An exact value here would be good.)
$$(An exact value here would be good.)$$

Finish the Proof.

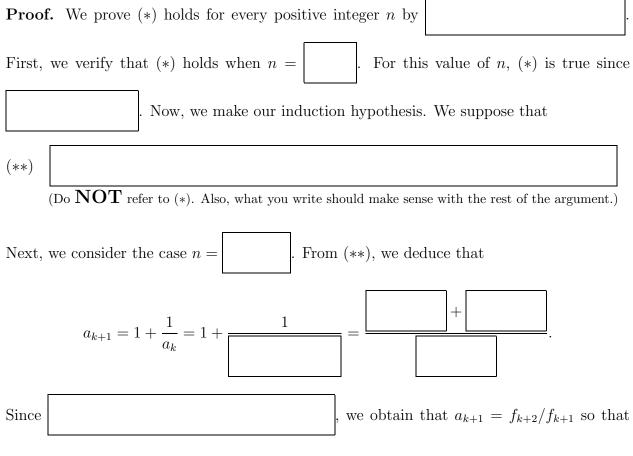
(6) Let

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{1}, \quad a_3 = 1 + \frac{1}{1 + \frac{1}{1}}, \quad a_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \dots$$

Thus, $a_1 = 1$, $a_2 = 2$, $a_3 = 3/2$, $a_4 = 5/3$, and in general $a_n = 1 + (1/a_{n-1})$ for each integer $n \ge 2$. Complete the proof below to show that

$$(*) a_n = \frac{f_{n+1}}{f_n}$$

holds for every positive integer n where f_n is the n^{th} Fibonacci number. Recall that $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for every integer $n \ge 1$. Fill in the boxes below appropriately. Do **NOT** refer to (*) in any of the boxes. Note that a proof should consist of complete English sentences.



(*) holds for n = k + 1. This proves (*) holds for every positive integer n by induction.