MATH 574, NOTES 2
PROOFS BY INDUCTION

Examples:

(1) The last digit of 6^n is a 6.

(2) What is the sum of the first n odd numbers? (Do this with a picture as well as induction.)

(3) Given the derivative rule $(fg)' = f'g + fg'$, explain how to compute $\frac{d}{dx}(x^n)$.

(4) For k an integer ≥ 2, show that

$$\cos(\pi/2^k) = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2}$$

where $k-1$ twos appear under the radicals.

(5) Show that $\int_0^{\pi/2} \cos^{2n} x \, dx = \frac{\pi}{2} \prod_{j=1}^{n} \left(\frac{2j-1}{2j} \right)$ for every integer $n \geq 1$.

(6) Suppose $a_1 = \sqrt{2}$, $a_2 = \sqrt{2}^{\sqrt{2}} = \sqrt{2}^{a_1}$, $a_3 = \sqrt{2}^{a_2}$, and so on. Then the values a_n are bounded.

(7) A positive integer which is one less than a multiple of 4 is divisible by a prime which is one less than a multiple of 4.

(8) All sheep are the same color.

(9) If f_k is the kth Fibonacci number (with $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, and so on), then show that f_n and f_{n+1} have no common prime divisor for every $n \geq 1$.

(10) Prove that $f_{n-1}f_{n+1} = f_n^2 + (-1)^n$ for every $n \geq 2$.

Homework:

(1) Prove that the last four digits of 625^n are given by 0625 for every integer $n \geq 2$.

(2) Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$. Try giving a proof with induction and a proof without induction.

(3) Show that $\int_0^{\pi/2} \cos^{2n+1} x \, dx = \prod_{j=1}^{n} \left(\frac{2j}{2j+1} \right)$ for every integer $n \geq 1$.

(4) Prove that f_{3n} is even and both f_{3n-1} and f_{3n-2} are odd for every $n \geq 1$. (Here f_k denotes the kth Fibonacci number.)

(5) The triangle inequality asserts that for any two real numbers x and y, the inequality $|x+y| \leq |x| + |y|$ holds. Using this, show that if $x_1, x_2, \ldots, x_n$ are n real numbers, then

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

(6) (a) Prove that for every integer $n \geq 1$, $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}$.

(b) Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ converge?

(7) Let D_n denote the number of ways to cover the squares of a $2 \times n$ board using plain dominos. Then it is easy to see that $D_1 = 1$, $D_2 = 2$, and $D_3 = 3$. Compute a few more values of D_n, guess an expression for the value of D_n, and use induction to prove you are right.

(8) (a) Let k be a positive integer. Using induction, prove that $\displaystyle\lim_{x \to \infty} \frac{(\log x)^k}{x} = 0$.

(b) Using a proof by contradiction and part (a), establish that there are infinitly many primes. (Hint: assume that there are exactly k primes and count the number of positive integers up to x.)