Math 574: Final Exam

Name ______________________________

Instructions and Point Values: Put your name in the space provided above. Check that your test contains 13 different pages including one blank page and this page. Work each problem below and show ALL of your work. Do NOT use a calculator.

There are 200 total points possible on this exam. The point total for each problem in each part is indicated below.

PART I:

Problem (1) is worth 14 points.

Problem (2) is worth 8 points.

Problem (3) is worth 14 points.

Problem (4) is worth 8 points.

Problem (5) is worth 14 points.

Problem (6) is worth 14 points.

Problem (7) is worth 14 points.

Problem (8) is worth 14 points.

PART II:

Problem (1) is worth 22 points.

Problem (2) is worth 22 points.

Problem (3) is worth 12 points.

Problem (4) is worth 22 points.

Problem (5) is worth 22 points.

PART I. Problems You've Seen (SHOW WORK!!)

(1) Suppose $a_0 = 2$, $a_1 = 3$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$. Find an explicit formula for a_n in terms of n.

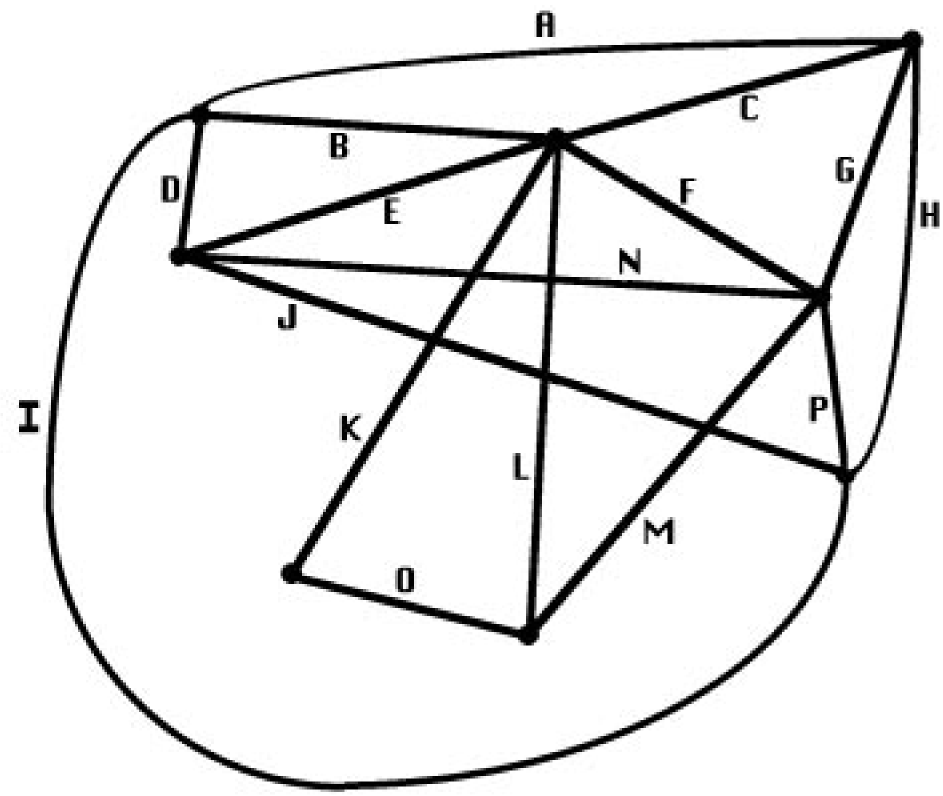
$a_n =$ [ ]

(2) If 3^k is the largest power of 3 dividing $200!$, then what is the value of k?

Answer: $\boxed{}$ (simplify your answer)

(3) Define $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2}^{a_n}$ for $n \geq 1$. Prove that $a_n \leq 2$ for all n.

(4) The graph below has an Euler path. The edges are labeled by the 16 letters A, B, C, . . . , O, and P. Write a list of letters that produces an Euler path (for example, ABCHI . . . would indicate a path that goes through the edges A, B, C, H, I, . . . in that order).



Answer:

(5) Given 6 points in the plane no three of which are collinear, suppose that the 15 line segments joining pairs of these points are each colored either red or blue. Using the pigeonhole principle, show that there must be a triangle formed with all of its edges the same color.

(6) Calculate $\sum_{k=0}^{n} k\binom{n}{k}$ in closed form. Show how the answer is derived (explain where the answer comes from).

Answer: [ ]

(7) How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 20$$

if each x_j is to be an integer from $\{0, 1, 2, \ldots, 20\}$? Show how the answer is derived. Don't just give a formula; if you want to use a formula, explain where the formula comes from.

Answer: [ ]

(8) Let A_n be the number of ways of covering a $2 \times n$ board using red 2×1 boards, blue 2×1 boards, green 2×1 boards, purple 1×2 boards, and yellow 1×2 boards. Thus, there are 3 ways of covering a 2×1 board (so $A_1 = 3$), namely by using one red 2×1 board or one blue 2×1 board or one green 2×1 board. Also, there are 13 ways to cover a 2×2 board (so $A_2 = 13$). For example, you could use a red 2×1 board followed by a green 2×1 board, or a green 2×1 board followed by a red 2×1 board, or a red 2×1 board followed by a red 2×1 board, or a purple 1×2 board above a yellow 1×2 board, or a yellow 1×2 board above a purple 1×2 board. That accounts for 5 of the 13 different coverings of a 2×2 board. Give an explicit formula for A_n in terms of n. In other words, find a recursion relationship that A_n satisfies and solve it.

$A_n =$ [ ]

PART II:

(1) The game of NIM is played with three stacks of coins having sizes 11, 21, and 30. Thus, two players take turns, each turn consisting of removing a positive number of coins from any one stack. The last person to remove a coin wins.

(a) Is it best to move first or second in this game? Justify your answer with correct work.

Answer: [ ] (answer one of "first" or "second")

(b) Suppose the first player decides to remove 2 coins from the stack of size 30. Then the stacks have sizes 11, 21, and 28. What is the best move for the second player to make in this situation? Justify your answer with correct work.

Answer: The second player should remove [ ] coins from the stack of size [ ].

(2) Prove that $\log_{10} 24$ is irrational.

(3) If the product

$$(x^2 - 3x + 1)^3(x - 2)^4(x + 1)^5(x^5 + x^3 - 2x^2 + 1)^2$$

is expanded, one gets the result

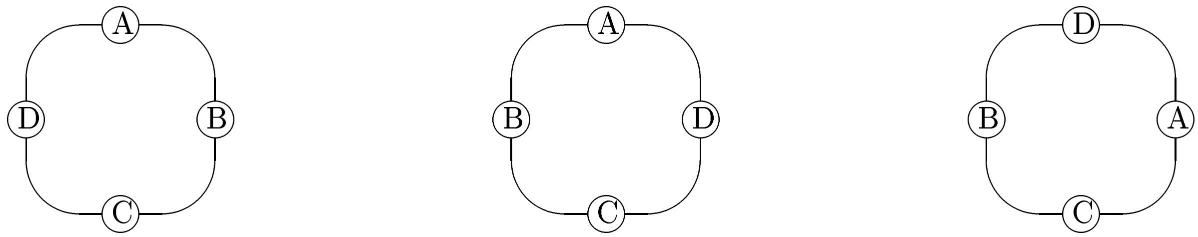
$$x^{25} - 12x^{24} + 53x^{23} - 91x^{22} - \cdots + 8x^2 - 96x + 16.$$

Only seven of the twenty-six terms are shown. What is the sum of all twenty-six coefficients (i.e., what is $1 - 12 + 53 - 91 - \cdots + 8 - 96 + 16$)? Explain your answer. Hint: This has something to do with our approaches for evaluating sums involving binomial coefficients.

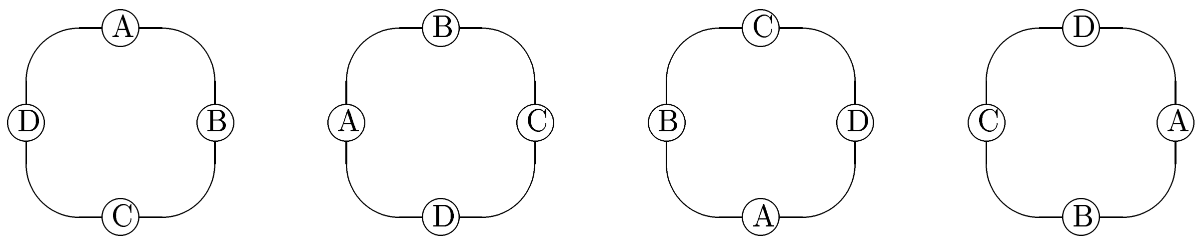
Answer: | |
|---|

Explanation:

(4) (a) The four letters A, B, C, and D can be arranged equally spaced around a circle (or oval) to form different arrangements. We consider, for example, the arrangements



as different. On the other hand, the arrangements given by



are all the same since one can be obtained from another by simply turning the circle around (for example, if the first circle is turned counter-clockwise 90°, then one gets the second circle). How many total different such arrangements are there for the four letters A, B, C, and D on the circle?

Answer:

(b) If instead the seven letters A, B, C, D, E, F, and G are arranged equally spaced around a circle, how many total different arrangements are there? Again, you should consider two arrangements the same if one can be obtained from the other by turning the circle.

Answer:

(5) Two players play a game on the board below as follows. Each person takes turns moving the letter "**S**" either downward (↓), to the left (←), or diagonally downward and to the left (↙). So each turn consists of moving *at least* one square in exactly one of these directions. For example, the first player may begin by moving to any one of 21 different squares, three of which are marked with an "**F**". The first person to place the "**S**" on the square marked with an "**E**" wins. How should the first player begin this game? Indicate the best first move by putting the letter "**A**" on the appropriate square (where the first player should make his first move). Justify your answer by labeling each square either "**B**" (for bad) or "**G**" (for good). For example, it would be good to move to the square marked with an "**E**" so put a "**G**" in that square.

	F								**S**
						F			
									F
E									